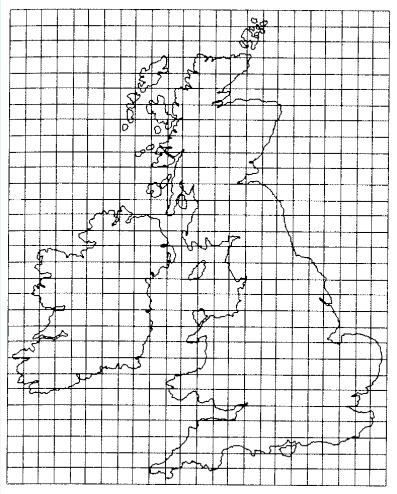
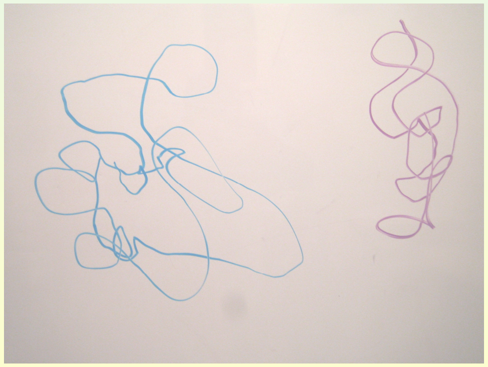
Session overview

- Hausdorff measure
- Definition of fractal and fractal dimension

Box-counting dimension



Chaos and Fractals: New Frontiers of Science, Peitgen, Jürgens, and Saupe, Springer-Verlag, 2004, p 205.

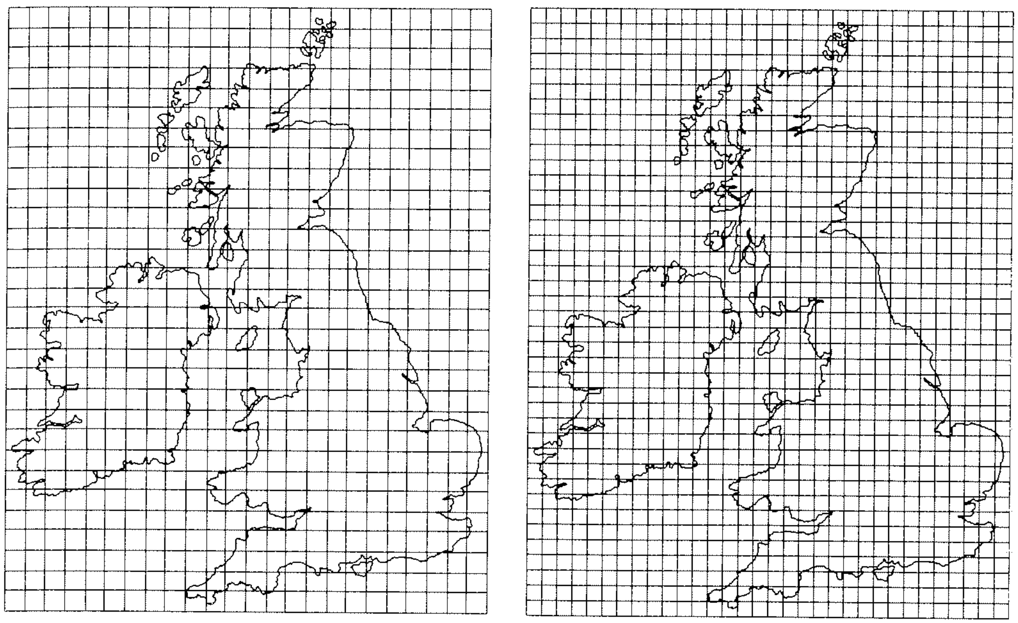


Elliot Boutell, age 3, 3/1/2008.

- Impose a grid of scale s.
- Count the number of cells, N(s), that contain the shape.
- Repeat with increasingly larger s.
- Plot (log(N(s)) as a function of log(1/s) (a "log-log" plot, used to reveal exponential relationships).
- The box-counting dimension, D_b, is the slope of the best-fit line between these points.
- Details in PJS 4.2

Box-counting dimension

$$m = \frac{\log 283 - \log 194}{\log 32 - \log 24} = \frac{2.45 - 2.29}{1.51 - 1.38} \approx 1.31$$



Fractal appearance;
Notice dimension > 1,
similar to Koch curve

Hausdorff measure (PJS 4.4)

- Let $S \subseteq \Re^n$ and let $B_i \subseteq \Re^n$
- Denote the diameter of B_i by $\text{diam}(B_i) = \sup \{ d(x,y): x, y \in B_i \}$, the greatest distance across B_i
- Define $H_\varepsilon^h(S) = \inf \{ \Sigma^i [\text{diam}(B_i)]^h : \text{diam}(B_i) \leq \varepsilon$ and $\{ B_i \}$ covers $S \}$
- Then, the Hausdorff h-dimensional measure of S is

$$H^h(S) = \lim_{\varepsilon \to 0} H_\varepsilon^h(S)$$

Example 1

- Let $S = \{ (1, 2), (2, 3), (-1, 2) \} \subseteq \Re^2$
- Let $B_1 = \{ (x, y): (x-1)^2 + (y-2)^2 < (\varepsilon/2)^2 \}$
- Let $B_2 = \{ (x, y): (x-2)^2 + (y-3)^2 < (\varepsilon/2)^2 \}$
- Let $B_3 = \{ (x, y): (x+1)^2 + (y-2)^2 < (\varepsilon/2)^2 \}$
- Think of $\{ B_i \}$ as a set of disks of diameter = ε
- As long as $\varepsilon < 1$, none of the disks centered at the points in S will overlap, and we require only 3 disks to cover S
- To compute $H^h(S)$ we have $0^h+0^h+0^h = 3$ (if $h = 0$) or 0 (if $h > 0$) (measure depends on dimension)

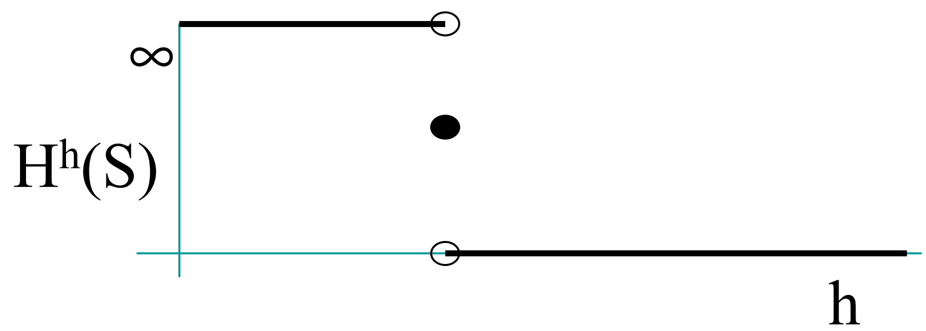
Example 2

- Let S = { [0, 2] × [0, 3] } ⊆ $\Re^2$
- Let B_i be a set of squares, each 1/n on a side, that covers S
- The diameter of each square is its diagonal, (√2)/n
- $H^h(S)$ is computed by:

$$\lim_{n\to\infty}\sum_{i=1}^{6n^2}\left(\frac{\sqrt{2}}{n}\right)^h = \lim_{n\to\infty} 6n^2\left(\frac{\sqrt{2}}{n}\right)^h = 6\left(\sqrt{2}\right)^h \lim_{n\to\infty} n^{2-h} = \begin{cases} \infty & h<2 \\ 6\left(\sqrt{2}\right)^h & h=2 \\ 0 & h>2 \end{cases}$$

Remark

- $H^h(S)$ is always decreasing
- It's graph typically looks like



- The Hausdorff dimension, D_H, is the critical value of h where the dimension jumps from ∞ to 0:

$$D_H(S) = \inf\{ h \mid H^h(S)=0\} = \sup\{ h \mid H^h(S)=\infty\}$$

Example 3

- Let S = the middle-thirds Cantor set
- At each stage, we require { B_i } to contain 2^n elements, each with diameter $(⅓)^n$ to cover the Cantor set without overdoing it
- Let $\varepsilon = (⅓)^n$ and let $\varepsilon \to 0$ by letting $n \to \infty$

$$H_\varepsilon^h(S) = \sum_{i=1}^{2^n} \left[\left(\frac{1}{3} \right)^n \right]^h = 2^n \left(\frac{1}{3^n} \right)^h$$

$$\therefore H^h(S) = \lim_{n \to \infty} 2^n \left(\frac{1}{3^n} \right)^h = \lim_{n \to \infty} \frac{2^n}{3^{nh}} = \lim_{n \to \infty} \left(\frac{2}{3^h} \right)^n$$

Example 3 (cont.)

- $H^0(S) = \lim_{n\to\infty} 2^n = \infty$
- $H^1(S) = \lim_{n\to\infty} (2/3)^n = 0$
- Note:
  - if $3^h > 2$, then $H^h(S) = 0$
  - if $3^h < 2$, then $H^h(S) = \infty$
  - if $3^h = 2$, then $H^h(S) = 1$
- $3^h = 2 \Rightarrow h = \log 2 / \log 3 = 0.631$
- To summarize, the Cantor set has:
  - Hausdorff 0-dimensional measure ∞
  - Hausdorff 1-dimensional measure 0
  - Hausdorff 0.631-dimensional measure 1
  - Summary: Hausdorff-dimension 0.631
  - self-similarity dimension 0.631
  - topological dimension 0
  - Lebesgue measure 0

The Koch snowflake paradox

- The Koch snowflake fits inside a circle even though it has infinite length
- This is not really a paradox if we think about length in the right way
- Length is usually thought of as L^1 length (or measure), which is equivalent to H^1 measure, infinity in this case
- Compute the Koch snowflake's Hausdorff dimension D_H (on quiz)

Equivalence of Hausdorff and self-similarity dimensions

- The Hausdorff dimension is equivalent to the self-similarity dimension for the case of self-similar fractals

Definition of fractal and fractal dimension

- Mandelbrot coined the term *fractal* in 1977
- A set $X \subseteq \Re^n$ is said to be a ***fractal*** if its Hausdorff dimension, D_H, strictly exceeds its topological dimension, D_T
- The number D_H is the ***fractal dimension*** of the set

Homework #1

- On the course web site
- Begin work on it in the remaining time we have in class