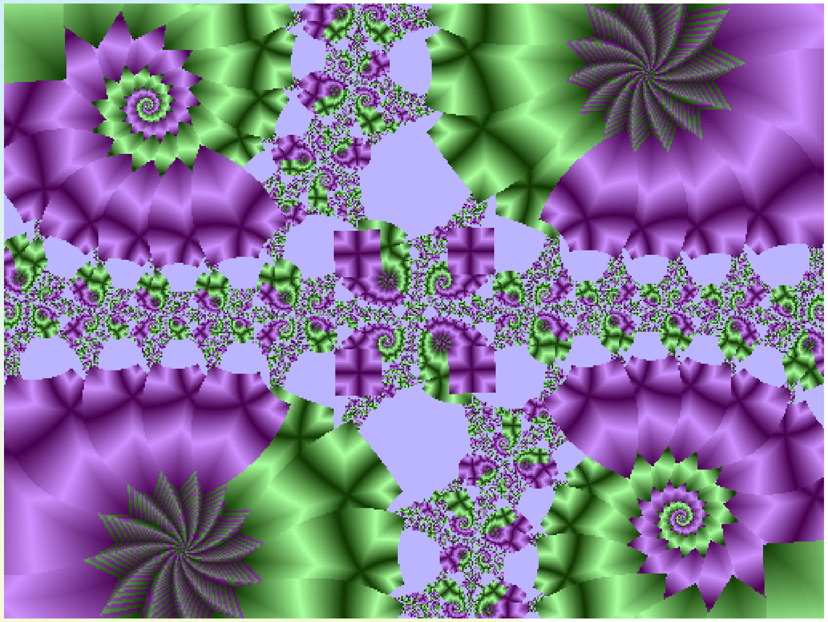
Session overview



- Measure and dimension

Announcements

- Homework 1 (posted by next class) due Tuesday in class.
- Project 1 due Tuesday night

Dimension is not adequate

- Dimension gives us a crude idea of the size of a set
- For example, the intervals $I_1=[0, 5]$ and $I_2=[2, 217]$ are both 1-dimensional
- However, I_1 has length 5 and I_2 has length 215

Measure

- Consider sets of discrete points, such as
  P_1={ (1, 2), (2, 3) } and
  P_2={ (2, 3), (1, 7), (1, -2), (7, 6), (5, 2) }
- dim (P_1) = dim (P_2) = 0
- However, the sets have different numbers of points
- Let the ***measure*** of a set be the number of points in the set, the measure of a line segment be its length, etc.
- We denote measure with the Greek letter μ
- Thus, $\mu(P_1)=2$, $\mu(P_2)=5$, $\mu(I_1)=5$, $\mu(I_2)=215$

Topological dimension

- The ***topological dimension*** of a set is our usual notion of dimension
- We all have the idea that a
  - point is 0-dimensional
  - line or curve is 1-dimensional
  - surface is 2-dimensional
  - region in space is 3-dimensional
- It is quite difficult to give a precise definition of topological dimension
- The usual way is done *inductively*, by defining what a 0-dimensional set is, then telling how to describe a 1-dimensional set from understanding a 0-dimensional set and continuing up through the non-negative integers
- See PJS 2.6

Topological dimension 0

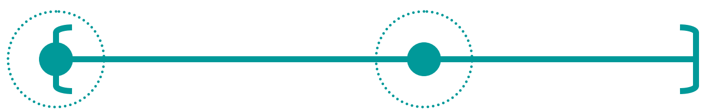
- A set, S, has topological dimension 0 if every point in S has arbitrarily small neighborhoods whose boundaries don't intersect the set
- We call such sets ***totally disconnected***

Topological dimension k

- A set, S, has topological dimension k (k=1, 2, 3, …) if k is the smallest natural number so that every point in S has arbitrarily small neighborhoods whose boundaries intersect S in a set of dimension k-1
- The underlined phrase isn't necessary unless you're dealing with fractals

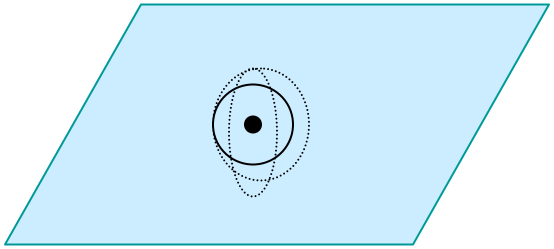
Example 1

- An interval has topological dimension 1, because every neighborhood of a point in an interval intersects at either 1 or 2 points
- In either case the intersection yields a 0-dimensional set



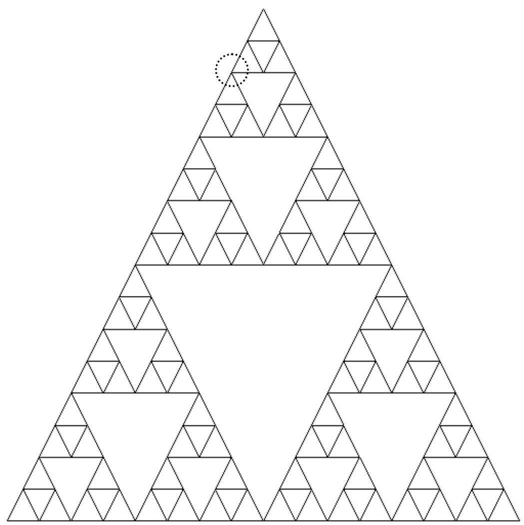
Example 2

- An surface has topological dimension 2, because every neighborhood of a point is a sphere that intersects the surface as either a closed curve in the interior or part of a curve on the boundary



Example 3

- You can always choose a neighborhood arbitrarily small that hits only __ or __ points, either way, the intersection is dimension 0
- So the Sierpinski triangle has topological dimension:
- 1



Examples 4 and 5

- The middle-thirds Cantor set has topological dimension 0. Why?
  - Consider any two points in the Cantor set
  - There must be a point between them that is not in it. Why?
  - Thus the Cantor set is totally disconnected
- What about the Koch curve?
- The Sierpinski *carpet*?

Different dimensions

- Sierpinski triangle
  - Topological dimension = 1
  - Fractal dimension = log 3 / log 2 = 1.585
- Cantor set
  - Topological dimension = 0
  - Fractal dimension = log 2 / log 3 = 0.631
- Koch curve
  - Topological dimension = 1
  - Fractal dimension = log 4 / log 3 = 1.262

Different sizes

- We can make many Sierpinski triangles, all of fractal dimension 1.585, but of obviously different sizes by choosing a larger boundary (triangle)
- Similarly, we could do the same for the Koch curve and the Cantor set
- To make sense of different sizes, we use the notion of measure

Lebesgue measure

- Pronounced: leh-BEG
- Capital I for **I**nterval…
- Let $I_i = [a_i, b_i]$, $a_i < b_i$ be intervals in $\Re$
- The Lebesgue (outer) measure of a set $S \subseteq \Re$ is

  $$L(S) = \inf \{ \Sigma_i |I_i| : \{I_i\} \text{ covers } S \}$$

  - inf = infimum = greatest lower bound
  - $|I_i|$ = length of I_i

Example 1

- Let $S = (0, 1] \cup [2, 7]$
- Let $I_1 = [-1, 2]$, $I_2 = [1, 3]$, $I_3 = [2, 8]$
- $\{ I_i \}$ covers S (i.e., $S \subseteq \cup I_i$)
- $|I_1| = 3$, $|I_2| = 2$, $|I_3| = 6$, $\Sigma|I_i| = 11$
- Since $L(S) = \inf \{ \Sigma|I_i| \}$, we know $L(S) \leq 11$
- Now let $I_1 = [0, 1]$ and $I_2 = [2, 7]$
- $\{I_1, I_2\}$ covers S and $\Sigma|I_i| = 1 + 5 = 6$
- We cannot create a better covering, so L(S) is 6

Example 2

- Let $X = \{ 1, 4, 7 \}$
- Let $I_1 = [1-\varepsilon/2, 1+ \varepsilon/2]$, $I_2 = [4-\varepsilon/2, 4+ \varepsilon/2]$, $I_3 = [7-\varepsilon/2, 7+ \varepsilon/2]$
- $\cup I_i$ covers X and $\Sigma|I_i| = 3\varepsilon$ for all $\varepsilon > 0$
- There is no minimum value for $\Sigma|I_i|$, but the greatest lower bound, or inf, is 0
- $L(X) = \inf \{ \Sigma|X_i| \} = \inf \{ 3\varepsilon \} = 0$

Example 3

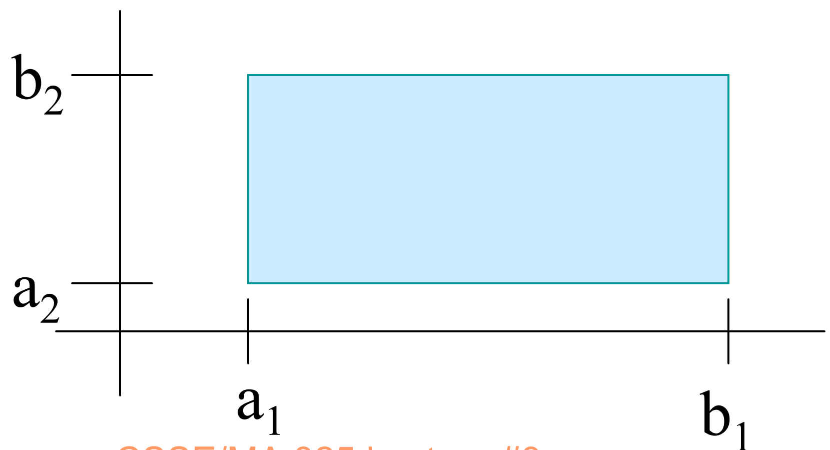
- Let X = middle-thirds Cantor set
- We can perform a perfect covering on level n with 2^n sub-intervals of width $(1/3)^n$
- Covering the n^{th} level covers X

$$L(X) = \inf\left\{\sum_{i=1}^{2^n}\left(\frac{1}{3}\right)^n\right\} = \inf\left\{2^n \cdot \left(\frac{1}{3^n}\right)\right\} = \lim_{n\to\infty}\left(\frac{2}{3}\right)^n = 0$$

- Remember this result! The Cantor set has no measure (in the Lebesgue sense)

Lebesgue measure on $\Re^n$

- Let $B_i = I_1^i \times I_2^i \times \ldots \times I_n^i$ be an n-dimensional box with each $I_j^i = [a_j, b_j]$, $a_j < b_j$
- In $\Re^2$, B_i is a rectangle with side lengths b_1-a_1 and b_2-a_2



Lebesgue measure on $\Re^n$ (cont.)

- Let $V(B_i)$ be the volume of B_i
- The n-dimensional Lebesgue measure of $S \subseteq \Re^n$ is

  $L^n(S) = \inf \{ \Sigma V(B_i): \{ B_i \} \text{ covers } S \}$
- It is easy to see from this definition that if S is any set of disjoint rectangles (and their interiors) in the plane, then the $L^2(S)$ = sum of the areas of the rectangles

Remark

- The L^2 measure says that to find the area of a region, find the minimum area of a bunch of rectangles that cover the region