ECE-320: Linear Control Systems
Homework 1

Due: Thursday March 11 *at the beginning of class*

1) The (one sided) Laplace transform is defined as

$$X(s) = \int_0^\infty x(t)e^{-st}dt$$

or the transform pair

$$x(t) \Leftrightarrow X(s)$$

a) By differentiating both sides of the above equation with respect to s, show that

$$-\frac{dX(s)}{ds} = \int_0^\infty tx(t)e^{-st}dt$$

$$\frac{d^2X(s)}{ds^2} = \int_0^\infty t^2x(t)e^{-st}dt$$

or

$$tx(t) \quad \Leftrightarrow \quad -\frac{dX(s)}{ds}$$

$$t^2x(t) \quad \Leftrightarrow \quad \frac{d^2X(s)}{ds^2}$$

b) By evaluating the integral, show that, if the real part of a is positive,

$$x(t) = e^{-at}u(t) \Leftrightarrow \frac{1}{s+a}$$

c) Combining parts (a) and (b), and a little bit of Maple free calculus, show that

$$x(t) = e^{-at}u(t) \quad \Leftrightarrow \quad \frac{1}{s+a}$$

$$tx(t) = te^{-at}u(t) \quad \Leftrightarrow \quad \frac{1}{(s+a)^2}$$

$$\frac{1}{2}t^2x(t) = \frac{1}{2}t^2e^{-at}u(t) \quad \Leftrightarrow \quad \frac{1}{(s+a)^3}$$

The moral of this problem is if a pole is repeated in the Laplace (s) domain, we just multiply by t in the time domain to get the shape of the time response. (There is still some scaling involved, but we are mostly concerned with the shape of the signal.)

2) Starting from the definition of the Laplace transform, show

$$x(t-t_0) \Leftrightarrow e^{-st_0} X(s)$$

Note that in this problem, we are assuming $x(t) = x(t)u(t)$ and that $x(t-t_0) = x(t-t_0)u(t-t_0)$. That is, we assume $x(t)$ is zero for $t < 0$ and $x(t-t_0)$ is zero for $t < t_0$.

The moral of this problem is that any time you see an e^{-st_0} in the Laplace domain, there is a delay, or transport lag, in the time domain.

3) Determine the impulse response of the following using partial fractions as necessary. You are expected to be able to do all of these with the Laplace transform Table in the notes, and the properties above. You may *check* your answers with Maple.

a) $H(s) = \dfrac{e^{-s}}{s^2 + 3}$ **b)** $H(s) = \dfrac{s+3}{(s+1)(s+2)}$ **c)** $H(s) = \dfrac{s}{(s+1)^2(s+3)}$

d) $H(s) = \dfrac{s+1}{s^2(s+2)}$ **e)** $H(s) = \dfrac{e^{-2s}(s+1)}{s(s+2)}$ **f)** $H(s) = \dfrac{1}{(s+1)(s+2)(s+3)(s+4)}$

4) (Model Matching) Consider the following closed loop system, with plant (the system we are trying to control) $G_p(s)$ and controller $G_c(s)$.
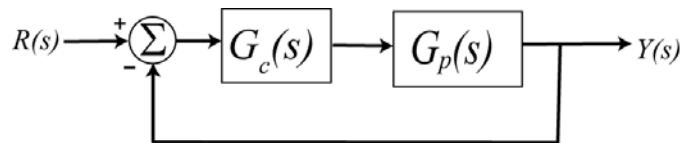
The transfer function for this closed loop system is

$$G_o(s) = \frac{Y(s)}{R(s)} = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)}$$

One way to choose the controller is to try and make your closed loop system match a transfer function that you choose (hence the name model matching). Let's assume that our **desired** closed loop transfer function $G_o(s)$, our plant can be written in terms of numerators and denominators as

$$G_o(s) = \frac{N_o(s)}{D_o(s)} \quad G_p(s) = \frac{N_p(s)}{D_p(s)}$$

Show that our controller is then $G_c(s) = \dfrac{N_o(s)D_p(s)}{N_p(s)[D_o(s) - N_o(s)]}$

Note that there are some restrictions here, in that for implementation purposes the controller must be stable, and it must be proper.

5) (Model Matching) For the following system, with plant $G_p(s) = \dfrac{1}{s+1}$, and controller $G_c(s)$

$R(s) \longrightarrow \boxed{\Sigma} \longrightarrow \boxed{G_c(s)} \longrightarrow \boxed{G_p(s)} \longrightarrow Y(s)$
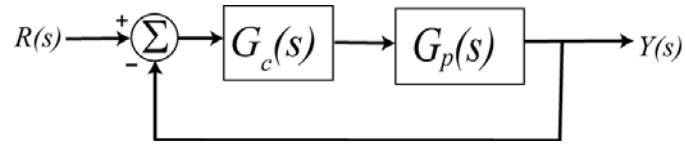
a) Using the results from problem 4, determine the controller so that the closed loop system matches a second order ITAE (**I**ntegral of **T**ime and **A**bsolute **E**rror) optimal system, i.e., so that the closed loop transfer function is

$$G_0(s) = \frac{\omega_0^2}{s^2 + 1.4\omega_0 s + \omega_0^2}$$

Ans. $G_c(s) = \dfrac{\omega_0^2(s+1)}{s(s+1.4\omega_0)}$, *note that there is a pole/zero cancellation between the controller and the*

plant and there is a pole at zero in the controller.

b) Determine the controller so that the closed loop system matches a third order **deadbeat** system, i.e., so that the closed loop transfer function is

$$G_0(s) = \frac{\omega_0^3}{s^3 + 1.90\omega_0 s^2 + 2.20\omega_0^2 s + \omega_0^3}$$

Ans. $G_c(s) = \dfrac{\omega_0^3(s+1)}{s(s^2 + 1.9\omega_0 s + 2.20\omega_0^2)}$, *note that there is a pole/zero cancellation between the controller*

*and the plant and there is a pole at zero in the controller.*

6) (Prelab/Matlab to be done individually)

a) Download the file homework1.rar from the class website and uncompress the files. The file **closedloop.mdl** is a Simulink file use the model a simple closed loop system. The file **closedloop_driver.m** is a Matlab file used to run the Simulink file **closedloop.mdl** and set various parameters. The file **state_model_1dof_rect** is a Matlab file that contains the parameter (a state variable model) for a one degree of freedom rectilinear system. In lab you will generate your own model for your own system to replace this one.

b) Start Matlab, and set the directory to the same folder these file are located in. Run the simulation, you should get results like the following:

$\omega_0 = 25$ rad/sec
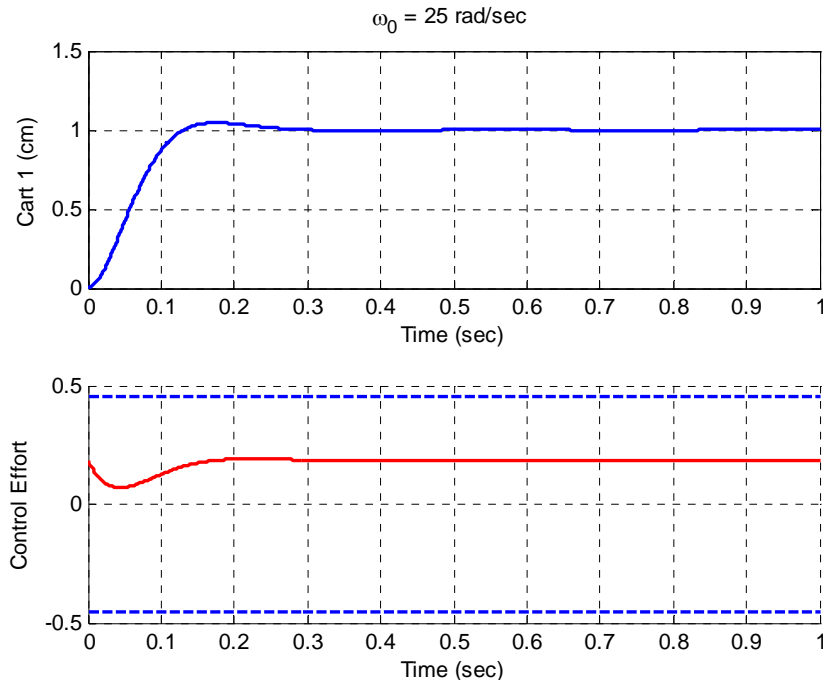
 If you cannot run the simulation, and you get some weird comment about "external mode", be sure you are in Normal mode (sometimes the newer versions of Matlab switch this….). See the following diagram for help
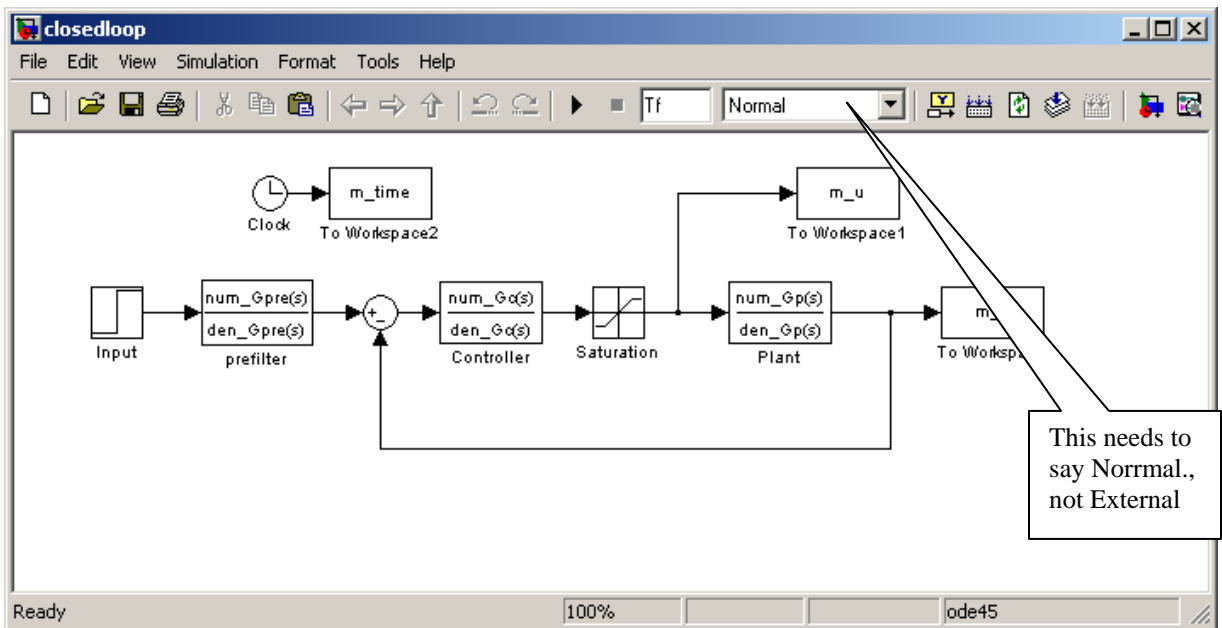
If your plot is not as smooth as that in the first figure, you need to select Simulation, then Configuration Parameters. You will get a screen basically like the following, be sure it is filled in as much as possible like this one, particularly the **Max step size** (set to 1e-3) , the **Relative tolerance** (set to 1e-3), and be sure it is using a **variable step** solver and the **ode45** solver in particular
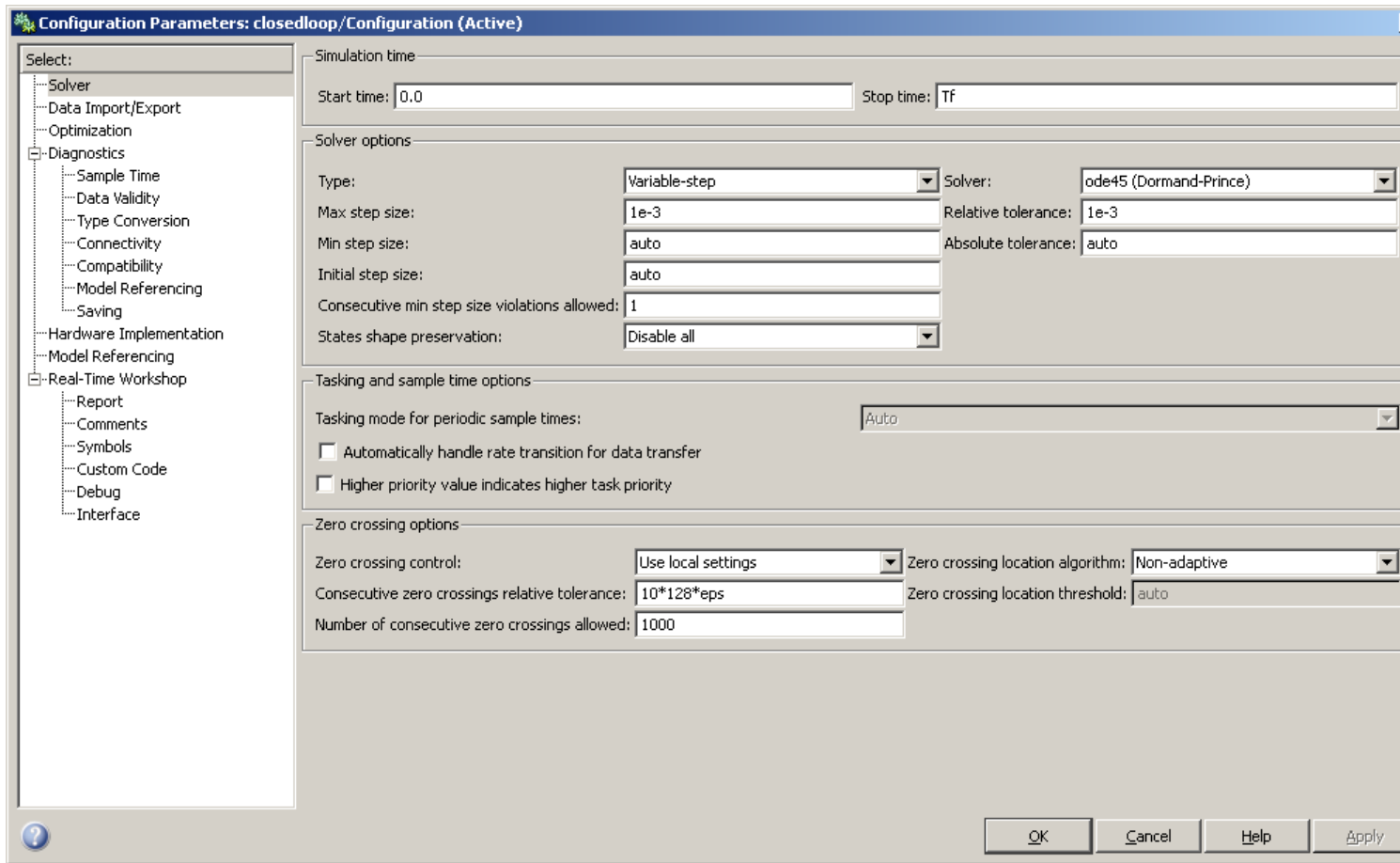
c) Set the parameter w0 to 100 and run the system. *Print out the simulation results and circle the location where the control effort is saturating Turn in this plot.* This is where we want the motor to provide more torque than it is capable of, and you will see that it is just as we turn the motor on. This is to be avoided!

d) Modify closedloop_driver.m so it is using the first deadbeat system. Run the simulation with w0 = 10, w0=30, and w0 = 50. Note that as w0 is increased, the system responds more quickly, but when w0 = 50 we are beginning to saturate the motor. *Turn in all three of these graphs.*