Calculators can only be used for simple calculations. Solving integrals, differential equations, systems of equations, etc. does not count as a simple calculation.

You must show your work to receive credit.

Problem 1 _________/16
Problem 2 _________/16
Problem 3 _________/14
Problem 4 _________/10
Problem 5 _________/12
Problem 6 _________/16
Problem 7-10 _________/16

Total _________
1) (16 points) Assume we have a first order system with the governing differential equation

\[ 0.2 \dot{y}(t) + y(t) = 3x(t) \]

The system has the initial value of 0.4, so \( y(0) = 0.4 \). The input to this system is

\[ x(t) = \begin{cases} 
0 & t < 0 \\
0.2 & 0 \leq t < 0.6 \\
0 & 0.6 \leq t 
\end{cases} \]

a) Determine the output of the system in each of the above time intervals. Simplify your final answer as much as possible and box it. Be sure to include the correct initial value in the first interval!

\[ y(t) = \left[ y(t_0) - Ka \right] e^{-k(t-t_0)}/c + Ka \]

\[ a) \quad 0 \leq t < 0.6 \quad y(0) = 0.4 \quad Ka = (3)(0.2) = 0.6 \quad c = 0.4 \]

\[ y(t) = \left[0.4 - 0.6\right] e^{-t/0.2} + 0.4 = \left[-0.2 e^{-t/0.2} + 0.4\right] = y(1.2) \]

\[ b) \quad 0.6 \leq t \quad y(0.6) = -0.2 e^{-3} + 0.6 = 0.59 \]

\[ Ka = (3)(0) = 0 \]

\[ y(t) = \left[0.59 - 0\right] e^{-\frac{(t-0.6)}{0.2}} + 0 = \left[0.59 e^{-\frac{(t-0.6)}{0.2}}\right] = y(1.4) \]
2) (16 points) For the following three differential equations, assume the input is $x(t) = 3u(t)$ (the input is equal to one for time greater than zero), and the initial conditions are $y(0) = y'(0) = 0$

Determine the solution to each of the following differential equations and put your final answer in a box. Be sure to use the initial conditions to solve for all unknowns. You must show all your work to receive credit, you cannot simply use the formula from problem 3.

a) $\ddot{y}(t) + 6\dot{y}(t) + 9y(t) = 9x(t)$

\[
\begin{align*}
\gamma^2 + 6\gamma + 9 &= 0 \\
(\gamma + 3)^2 &= 0 \\
y_0 &= 0 \\
y_1 &= 3
\end{align*}
\]

$y(t) = 3 + c_1 e^{-3t} + c_2 t e^{-3t}$

$y(0) = 0 = 3 + c_1$ \hspace{1cm} $c_1 = -3$

$\dot{y}(t) = 0 - 3c_1 e^{-3t} + c_2 e^{-3t} - 3c_2 t e^{-3t}$

$\dot{y}(0) = -3c_1 + c_2 = 0$ \hspace{1cm} $c_2 = 3c_1 = -9 = c_2$

$y(t) = 3 - 3e^{-3t} - 9te^{-3t}$

b) $\ddot{y}(t) + 4\dot{y}(t) + 13y(t) = 26x(t)$

\[
\begin{align*}
\gamma^2 + 4\gamma + 13 &= 0 \\
(\gamma + 2)^2 + 3^2 &= 0 \\
\gamma &= -2 \pm j3
\end{align*}
\]

$y(t) = c_1 e^{-2t} \sin(3t + \theta)$

$y(0) = 0 = c_1 + c_2 \sin(\theta)$

$\dot{y}(t) = -2c_1 e^{-2t} \sin(3t + \theta) + 3c_2 e^{-2t} \cos(3t + \theta)$

$\dot{y}(0) = 0 = -2c_1 \sin(\theta) + 3c_2 \cos(\theta)$

$tan(\theta) = \frac{3}{2}$ \hspace{1cm} $\theta = 56.31^\circ$

$C = \frac{-6}{sin(56.31^\circ)} = -7.21 = C$

$y(t) = (6 - 7.21 e^{-2t} \sin(3t + 56.31^\circ))$
3) (14 points) The form of the under damped \((0 < \zeta < 1)\) solution to the second order differential equation

\[
\ddot{y}(t) + 2\zeta \omega_n \dot{y}(t) + \omega_n^2 y(t) = K \omega_n^2 x(t)
\]

for a step input \(x(t) = Au(t)\) is

\[
y(t) = KA + ce^{-\zeta \omega_n t} \sin(\omega_d t + \phi)
\]

where \(c\) and \(\phi\) are constants to be determined and the damped frequency \(\omega_d = \omega_n \sqrt{1 - \zeta^2}\)

a) Using the initial condition \(\dot{y}(0) = 0\) show that \(\tan(\phi) = \frac{\sqrt{1 - \zeta^2}}{\zeta}\)

\[
\dot{y}(t) = 0 = -\omega_n c e^{-\omega_n t} \sin(\omega_d t + \phi) + \omega_n c e^{-\omega_n t} \cos(\omega_d t + \phi)
\]

\[
\dot{y}(0) = 0 = -\omega_n c \sin(\phi) + \omega_n c \cos(\phi)
\]

\[
\tan(\phi) = \frac{\omega_n c \cos(\phi) - \omega_n c \sin(\phi)}{\omega_n c \sin(\phi)} = \frac{\omega_n \sqrt{1 - \zeta^2}}{\omega_n \zeta}
\]

\[
\tan(\phi) = \frac{\sqrt{1 - \zeta^2}}{\zeta}
\]

b) We can express the relationship in part a using the following triangle. Fill in the blanks and then use this triangle determine an expression for \(\sin(\phi)\).

\[
\begin{align*}
r &= 1 \\
x &= \ldots \\
y &= \sqrt{1 - \zeta^2} \\
\sin(\phi) &= \sqrt{1 - \zeta^2}
\end{align*}
\]

\(r^2 = y^2 + x^2 = 1 - \zeta^2 + \zeta^2 = 1\)

c) Use your answer to part b, and the initial condition \(\dot{y}(0) = 0\) to determine the remaining unknown constant, and write out the complete solution for \(y(t)\).

\[
y(0) = 0 = KA + c \sin(\phi) = KA + CV_1 \sqrt{1 - \zeta^2} \Rightarrow c = -\frac{KA}{V_1 \sqrt{1 - \zeta^2}}
\]

\[
y(t) = KA \left[ 1 - e^{-\omega_n t} \frac{\sin(\omega_d t + \phi)}{V_1 \sqrt{1 - \zeta^2}} \right]
\]
4) (10 points) For the following first order differential equation,

\[ 3\dot{y}(t) + y(t) = x^2(t) \]

determine an expression for the output assuming \( t_0 = 0 \) and \( y(t_0) = y(0) = 1 \).

\[
\begin{align*}
\dot{y}(t) + \frac{1}{3} y(t) &= \frac{1}{3} x^2(t) \\
\frac{d}{dt} \left[ y(t) e^{t/3} \right] &= \frac{1}{3} e^{t/3} x^2(t) \\
y(t) e^{t/3} - y(t_0) e^{t_0/3} &= \int_{t_0}^{t} e^{\gamma_3 \lambda} x^2(\lambda) d\lambda \\
y(t) e^{t/3} &= \frac{1}{3} \int_{0}^{t} e^{\gamma_3 \lambda} x^2(\lambda) d\lambda \\
y(t) &= e^{-t/3} + \frac{1}{3} e^{-t/3} \int_{0}^{t} e^{\gamma_3 \lambda} x^2(\lambda) d\lambda
\end{align*}
\]
5) **(12 points)** For the following two op-amps circuits, we can write \( v_{\text{out}}(t) = G \ v_{\text{in}}(t) \). Determine the value of \( G \) for each circuit.

\[
V^* = \frac{V_{\text{in}}}{R_a + R_b} \quad \frac{V^*}{R_d} + \frac{V_{\text{out}}}{R_e} = 0 \quad V_{\text{out}} = -\frac{R_e}{R_d} V^* \\
G = -\frac{R_e}{R_d} \frac{R_b}{R_a + R_b}
\]

\[
V^* = \frac{V_{\text{in}}}{R_a + R_b} \\
V_{\text{out}} = \frac{R_b}{R_g + R_p} V_{\text{in}} \\
V_{\text{out}} = \left( \frac{R_g + R_p}{R_g} \right) V_{\text{in}} \\
G = \frac{R_g + R_p}{R_g} \frac{R_b}{R_a + R_b}
\]
6) (16 points) For the second order circuit below,

![Circuit Diagram]

Derive the governing second order differential equation for the output $y(t)$ and input $x(t)$. You do not need to put the equation in standard form.

\[ x(t) = C \left( \frac{d^2 y(t)}{dt^2} + y(t) \right) \]

\[ V(t) = y(t) R - L \frac{d y(t)}{dt} = 0 \]

\[ V(t) = y(t) R + L \frac{d^2 y(t)}{dt^2} \]

\[ x(t) = C \left[ \frac{d^2 y(t)}{dt^2} R + L \frac{d y(t)}{dt} \right] + y(t) \]

\[ x(t) = LC \frac{d^2 y(t)}{dt^2} + RC \frac{d y(t)}{dt} + y(t) \]
Problems 7-10, 4 points each (16 points)

For problems 7 and 8, refer to the following graph showing the input and output of a first order system. For this system the input is a step of amplitude 1.5.

7) What is the static gain? \( K(1,5) = 3 \quad \boxed{K = 2} \)

8) What is the time constant? \( T = 4 \text{ sec} \quad \boxed{T = 0.25 \text{ sec}} \)
For problems 9 and 10, refer to the following graph showing the input and output of a second order system. For this system the input is a step of amplitude 4.

9) What is the static gain of the system?

\[ K(4) = 6 \]
\[ K = \frac{6}{4} = 1.5 = K \]

10) What is the percent overshoot?

\[ \frac{9 - 4}{4} \times 100\% = 50\% \]