ECE-205
Exam 1
Spring-2011

Calculators can only be used for simple calculations. Solving integrals, differential equations, systems of equations, etc. does not count as a simple calculation.

You must show your work to receive credit.

Problem 1 ________/20
Problem 2 ________/30
Problem 3 ________/20
Problem 4 ________/10
Problem 5 ________/10
Problem 6 ________/10

Total ____________
1) (20 points) Assume we have a first order system with the governing differential equation

\[ 0.1 \dot{y}(t) + y(t) = 2x(t) \]

The system has the initial value of 0, so \( y(0) = 0 \). The input to this system is

\[ x(t) = \begin{cases} 
0 & t < 0 \\
-1 & 0 \leq t < 0.2 \\
-2 & 0.2 \leq t < 0.5 \\
3 & 0.5 < t 
\end{cases} \]

Determine the output of the system in each of the above time intervals. Simplify your final answer as much as possible and box it. Be sure to include the correct initial value in the first interval!

\[ y(t) = [y(t_0) - y(\infty)] e^{-(t-t_0)/\tau} + y(\infty) \quad \kappa = 2 \quad \tau = 0.1 \]

1. \( A = -1 \quad KA = y(\infty) = -2 \quad t_0 = 0 \quad y(t_0) = 0 \)

\[ y(t) = [0 - (-2)] e^{-(t-0)/0.1} + (-2) = \frac{2 e^{-t/0.1} - 2 = y(t)}{0 \leq t \leq 0.2} \]

\[ y(0.2) = 2 e^{0.2/0.1} - 2 = -1.7293 \]

2. \( A = -2 \quad KA = y(\infty) = -4 \quad t_0 = 0.2 \quad y(t_0) = -1.7293 \)

\[ y(t) = [-1.7293 - (-4)] e^{-(t-0.2)/0.1} + (-4) = \frac{2.2707 e^{-(t-0.2)/0.1} - 4 = y(t)}{0.2 \leq t \leq 0.5} \]

\[ y(0.5) = 2.2707 e^{0.3/0.1} - 4 = -3.8869 \]

3. \( A = 3 \quad KA = y(\infty) = 6 \quad t_0 = 0.5 \quad y(t_0) = -3.8869 \)

\[ y(t) = [-3.8869 - 6] e^{-(t-0.5)/0.1} + 6 = \frac{-9.8869 e^{-(t-0.5)/0.1} + 6 = y(t)}{t \geq 0.5} \]
2) (30 points) For the following three differential equations, assume the input is \( x(t) = 4u(t) \) (the input is equal to four for time greater than zero), and the initial conditions are \( y(0) = y(0) = 0 \)

Determine the solution to each of the following differential equations and put your final answer in a box. Be sure to use the initial conditions to solve for all unknowns. You must show all your work to receive credit.

a) \( \dot{y}(t) + 5\ddot{y}(t) + 4y(t) = 2x(t) \)  
\[ \begin{align*} 
\dot{y}(0) &= 2, \quad \ddot{y}(0) = 2, \quad (r+1)(r+4) = 0 \\
\dot{y}(t) &= c_1 e^{-t} + c_2 e^{4t} + 2 \\
y(0) &= c_1 + c_2 + 2 = 0 \\
\ddot{y}(t) &= -c_1 e^{-t} - 4c_2 e^{4t} \\
\ddot{y}(0) &= -c_1 - 4c_2 = 0 \\
\text{adding:} \quad -3c_2 + 2 = 0 \\
c_2 &= \frac{2}{3} \\
\therefore c_1 &= -\frac{8}{3} \quad (c_1, c_2) = \left(-\frac{8}{3}, \frac{2}{3}\right) \\
\boxed{y(t) = -\frac{8}{3} e^{-t} + \frac{2}{3} e^{4t} + 2} 
\end{align*} \]

b) \( \dot{y}(t) + 6\ddot{y}(t) + 9y(t) = 9x(t) \)  
\[ \begin{align*} 
\dot{y}(0) &= 9, \quad \ddot{y}(0) = 4, \quad (r+3)^2 = 0 \\
y(0) &= c_1 + c_2 = 0 \\
\ddot{y}(t) &= -3c_1 e^{-3t} + c_2 e^{-3t} - 3t c_2 e^{-3t} \\
\ddot{y}(0) &= 0 = -3c_1 + c_2 \\
c_2 &= 3c_1 = -12 = c_2 \\
\boxed{y(t) = -4 e^{-3t} - 12t e^{-3t} + 4} 
\end{align*} \]

c) \( \dot{y}(t) + 4\ddot{y}(t) + 13y(t) = 6.5x(t) \)  
\[ \begin{align*} 
\dot{y}(0) &= 6.5, \quad \ddot{y}(0) = 2, \quad r^2 + 4r + 13 = 0 \\
r &= -\frac{4 \pm \sqrt{16 - 52}}{2} = -2 \pm 3i \\
y(0) &= c \sin(3t + \Theta) + 2 \\
\dot{y}(t) &= -2 c e^{-2t} \sin(3t + \Theta) + 3 c e^{-2t} \cos(3t + \Theta) \\
\dot{y}(0) &= 0 = -2 \frac{\sin(\Theta)}{\sin(3\Theta)} \\
\therefore \sin(3\Theta) &= \frac{3}{2} = \frac{3\sin(\Theta)}{2} \Rightarrow 3\Theta = \sin^{-1}(\frac{3}{2}) \Rightarrow \Theta = \frac{\sin^{-1}(\frac{3}{2})}{3} = -2.403 \rangle \\
\boxed{y(t) = 2 - 2.403 e^{-2t} \sin(3t + 56.3\circ)}} 
\end{align*} \]
3) **(20 points)** For a simple series RC circuit the response of the system when the input is a unit step is

\[ y(t) = 1 - e^{-t/RC} = 1 - e^{-t/\tau} \]

The 10-90% rise time, \( \tau_r \), as shown below. The rise time is simply the amount of time necessary for the output to rise from 10% to 90% of its final value. Show that for this system the rise time is given by \( \tau_r = \tau \ln(9) \)

\[ y(t_{10}) = 0.1 = 1 - e^{-t_{10}/\tau} \]
\[ y(t_{90}) = 0.9 = 1 - e^{-t_{90}/\tau} \]
\[ e^{-t_{10}/\tau} = 0.1 \]
\[ e^{-t_{90}/\tau} = 0.9 \]

\[ e^{-t_{10}/\tau} = e^{(t_{90} - t_{10})/\tau} = e^{v/\tau} = q \]

\[ \frac{t_r}{\tau} = \ln q \]

\[ t_r = \tau \ln q \]
4) (10 points) The initial portion of the response of a first order circuit to a step input is shown in the figure below. The steady state value of the output is 2.0 volts. Using this data, estimate the time constant of the system. Show your work or you receive no credit! (This requires very little work, but be sure to show it.)

\[ y(t) = 0.98 \times \left(\frac{2}{1}\right) = 1.96 \] 

This happens at \( T_s = 4.2 \approx 2 \)

\[ 2 \times 0.5 \, \text{sec} = 1 \]
5) (10 points) For the following circuit, determine an expression for the time constant and the static gain. *(This can pretty much be done by inspection.)*

\[ R_{th} = R_a + R_b \]

\[ \tau = \frac{L}{R_{th}} \]

\[ K = 1 \]
6) (10 points) Using integrating factors, solve the following differential equation for an arbitrary input \( x(t) \) for the initial conditions \( t_0 = 0 \) and \( y(0) = 1 \). Note that your solution will involve an integral, since you do not know what the input \( x(t) \) is. \textbf{Show your work or you receive no credit!}

\[
\frac{1}{2} \dot{y}(t) - ty(t) = 3x(t)
\]

\[
y'(t) - 2ty(t) = 6x(t)
\]

\[
dt \left[ y(t)e^{-t^2} \right] = (6e^{-t^2}x(t))
\]

\[
y(t)e^{-t^2} - y(0)e^{-t_0^2} = \int_{t_0}^{t} 6e^{-\lambda^2}x(\lambda)\,d\lambda
\]

\[
y(t)e^{-t^2} = 1 + 6\int_{0}^{t} e^{-\lambda^2}x(\lambda)\,d\lambda
\]

\[
y(t) = e^{t^2} + 6e^{t^2}\int_{0}^{t} e^{-\lambda^2}x(\lambda)\,d\lambda
\]