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# A jumping cylinder on an inclined plane

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## Abstract

The problem of a cylinder of mass  $m$  and radius  $r$ , with its centre of mass out of the cylinder's axis, rolling on an inclined plane that makes an angle  $\alpha$  with respect to the horizontal, is analysed. The equation of motion is partially solved to obtain the site where the cylinder loses contact with the inclined plane (jumps). Several simplifications are made: the analysed system consists of an homogeneous disc with a one-dimensional straight line mass parallel to the disc axis at a distance  $y < r$  of the centre of the cylinder. To compare our results with experimental data, we use a styrofoam cylinder to which a long brass rod is embedded parallel to the disc axis at a distance  $y < r$  from it, so the centre of mass lies at a distance  $d$  from the centre of the cylinder. Then the disc rolls without slipping on a long wooden ramp inclined at  $15^\circ$ ,  $30^\circ$  and  $45^\circ$  with respect to the horizontal. To determine the jumping site, the movements are recorded with a high-speed video camera (Casio EX ZR100) at 240 and 480 frames per second. The experimental results agree well with the theoretical predictions.

 Online supplementary data available from [stacks.iop.org/EJP/33/1359/mmedia](http://stacks.iop.org/EJP/33/1359/mmedia)

(Some figures may appear in colour only in the online journal)

## 1. Introduction

The motion of a homogeneous cylinder rolling without slipping on a horizontal surface and on an inclined plane is a typical problem in most mechanics textbooks, but only a few address the same problem when the centre of mass of the cylinder is out of its axis; actually, we only know of two examples where this case is put forward as an end-of-the-chapter exercise [1, 2]. Moreover, we have only found a few references that deal with similar movements for symmetrical inhomogeneous bodies [3–9]. In [3], the Lagrangian formulation is used to describe the motion of two cylinders moving in a horizontal plane. References [4–7] use a cumbersome torque analysis in the solution of the movement. None of them establishes the jumping site of an inhomogeneous cylinder. The purpose of this paper is to put forward a common type of movement forgotten in most undergraduate and postgraduate textbooks.

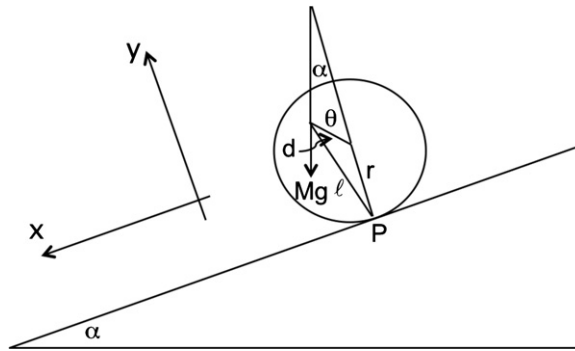


Figure 1. The rolling cylinder at an arbitrary position.

## 2. Theoretical solution

We analyse a simplified model of the inhomogeneous cylinder, in which we take the centre of mass to be out of the geometrical centre of the cylinder. The mechanical system considered, depicted in figure 1, consists of a homogeneous disc of mass  $m_1$ , with a line mass  $m_2$  located at some distance from the centre of the cylinder. The physical quantities of interest in this development are (figure 1) as follows.

- $\alpha$  is the inclined plane angle with respect to the horizontal;
- $r$  is the cylinder radius;
- $M = m_1 + m_2$  is the total mass of the system;
- $d$  is the distance from the centre of the cylinder to the centre of mass;
- $\theta$  is the rotation angle;
- $P$  is the instantaneous axis of rotation;
- $\ell$  is the distance from  $P$  to the centre of mass;
- $I_{CM} = \frac{1}{2}m_1r^2 + \frac{m_1M}{m_2}d^2$  is the moment of inertia of the cylinder with respect to a perpendicular axis passing through the centre of mass;
- $I_P = I_{CM} + M\ell^2$  is the moment of inertia with respect to  $P$ .

The simplest way to partially solve the equation of motion is through energy conservation. After the cylinder has rolled a distance  $x = r\theta$  along the inclined plane, its geometrical centre has descended a distance  $h_1 = r\theta \sin \alpha$  and the centre of mass has descended a distance  $h_2 = d - d \cos \theta \cos \alpha$ , so the change  $\Delta U$  in the potential energy is

$$\Delta U = -Mg[r\theta \sin \alpha + d(1 - \cos \theta) \cos \alpha], \quad (1)$$

where  $g$  is the value of the acceleration of gravity.

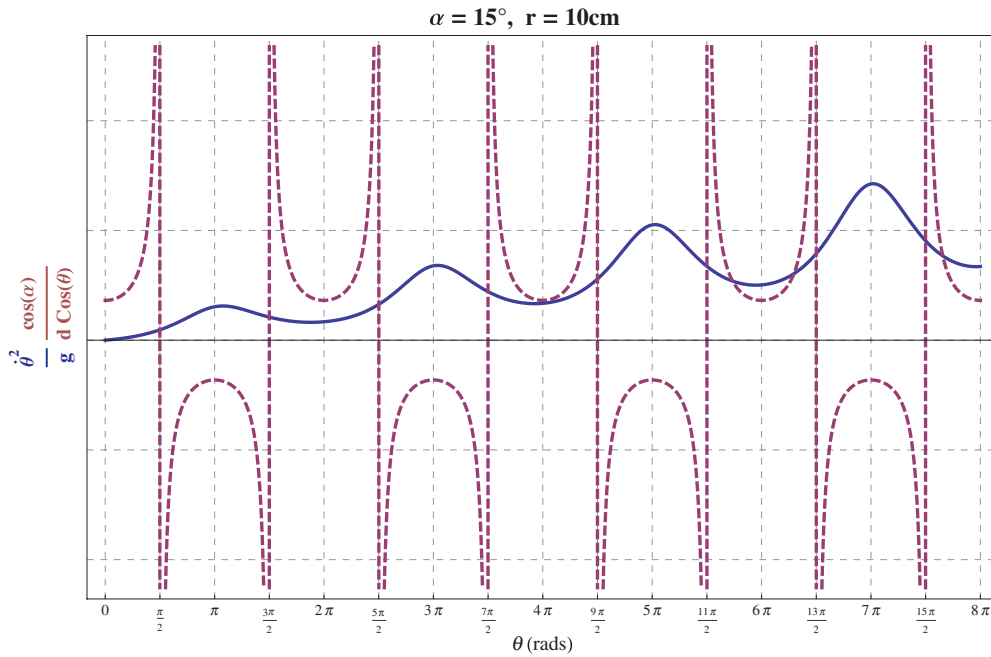
If the initial conditions are such that the cylinder is at rest and the centre of mass lies on a line perpendicular to the inclined plane, then the change  $\Delta K$  in the kinetic energy of the disc, after it has rotated an angle  $\theta$ , is

$$\Delta K = \frac{1}{2}I_P\dot{\theta}^2, \quad (2)$$

where  $I_P$  is the cylinder's moment of inertia with respect to the instantaneous axis of rotation  $P$ .

Taking a reference frame in which the  $x$ -axis is along the inclined plane, the  $y$ -axis is perpendicular to the same plane and the  $z$ -axis is perpendicular to the  $x$ - $y$  plane, the following relation holds between the different quantities involved in the problem (figure 1)

$$\ell = \mathbf{r} + \mathbf{d} = rd \sin \theta + j(r + d \cos \theta), \quad (3)$$



**Figure 2.** Plot of  $\dot{\theta}^2/g$  versus  $\cos \alpha/d \cos \theta$  for  $\alpha = 15^\circ$ . The first intersection is at  $\theta = 18.01$  rad, so the cylinder jumps at 180.1 cm.

where  $\ell$  is the vector from the centre of mass to the instantaneous axis of rotation  $P$  and  $\theta$  is the rotation angle. The magnitude of  $\ell$  is

$$\ell = \sqrt{r^2 + d^2 + 2rd \cos \theta}. \tag{4}$$

Combining equations (1) and (2) one obtains

$$\dot{\theta}^2 = 2Mg \frac{d(1 - \cos \theta) \cos \alpha + r\theta \sin \alpha}{I_{CM} + M(r^2 + d^2 + 2rd \cos \theta)}. \tag{5}$$

The condition the cylinder must satisfy in order to lose contact with the inclined plane is that the normal to the inclined plane component of the centrifugal force,

$$\mathbf{F}_C = -M\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{d}) = M(id \sin \theta + jd \cos \theta)\dot{\theta}^2, \tag{6}$$

equals the normal component of the cylinder total weight, that is,

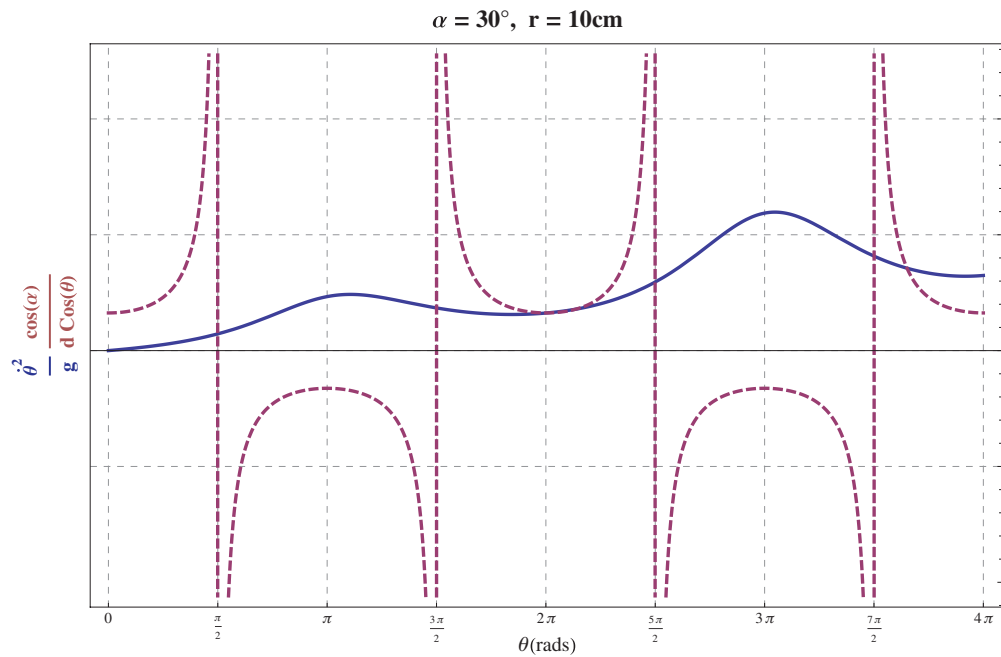
$$Md\dot{\theta}^2 \cos \theta = Mg \cos \alpha, \tag{7}$$

from which the following expression can be obtained, using equation (5):

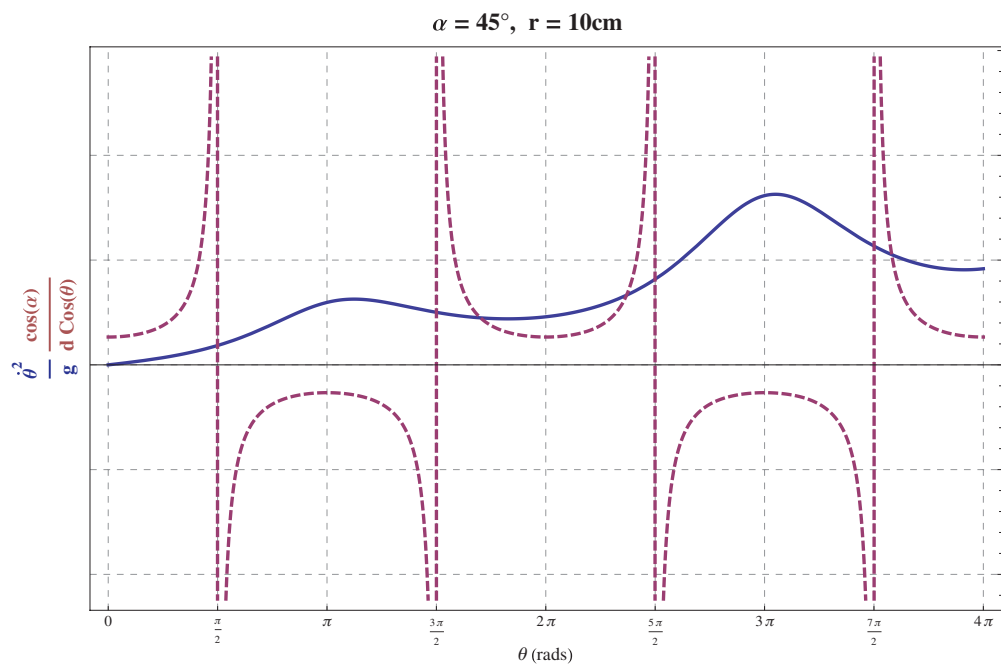
$$2M \left\{ \frac{d(1 - \cos \theta) \cos \alpha + r\theta \sin \alpha}{I_{CM} + M(r^2 + d^2 + 2rd \cos \theta)} \right\} = \frac{\cos \alpha}{d \cos \theta}. \tag{8}$$

A simple way to arrive at a solution of this transcendental equation is plotting both sides and looking for the first intersection of the resulting curves. The results for  $15^\circ$ ,  $30^\circ$  and  $45^\circ$  are shown in figures 2, 3 and 4, respectively, where the dashed lines correspond to the right-hand side of equation (6), while the continuous one corresponds to its left-hand side.

It is interesting to note the oscillation of the kinetic energy (proportional to the left-hand side of equation (8)) due to the centrifugal force effect produced at different positions of the centre of mass of the cylinder.



**Figure 3.** Plot of  $\dot{\theta}^2/g$  versus  $\cos \alpha/d \cos \theta$  for  $\alpha = 30^\circ$ . The first intersection is at  $\theta = 11.46$  rad, so the cylinder jumps at 114.6 cm.



**Figure 4.** Plot of  $\dot{\theta}^2/g$  versus  $\cos \alpha/d \cos \theta$  for  $\alpha = 45^\circ$ . The first intersection is at  $\theta = 5.34$  rad, so the cylinder jumps at 53.4 cm.

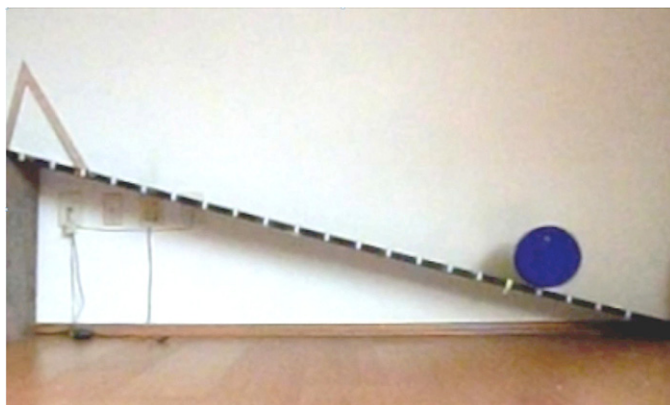


Figure 5. Photogram at the jumping site for an inclined plane at  $\alpha = 15^\circ$ .

### 3. Experimental results

To compare our theoretical results with experimental data, we use a styrofoam cylinder of radius  $r = 10.00 \pm 0.05$  cm, height  $h = 5.55 \pm 0.05$  cm and mass  $m_1 = 34.20 \pm 0.05$  g, to which a  $9.50 \pm 0.01$  mm diameter and  $5.10 \pm 0.01$  cm long brass rod of mass  $m_2 = 32.10 \pm 0.05$  g was embedded parallel to the disc axis at a distance  $y = 5.50 \pm 0.05$  cm from it, so the centre of mass lies at a distance  $d = 2.65 \pm 0.09$  cm from the centre of the cylinder. Then the disc rolls on a long wooden ramp inclined at  $15^\circ$ ,  $30^\circ$  and  $45^\circ$  with respect to the horizontal, by means of squares with such corresponding angles. To determine the jumping site, the motion was recorded with a high-speed video camera (Casio EX ZR100) at 240 and 480 frames per second (fps), with corresponding resolutions of  $432 \times 320$  and  $224 \times 160$  pixels. For the measurements, we used the higher resolution (240 fps) videos (available from [stacks.iop.org/EJP/33/1359/mmedia](https://stacks.iop.org/EJP/33/1359/mmedia)), which leads to an uncertainty in the position of the jumping site of the cylinder of about 0.5 cm. The acceleration of gravity in Mexico City is  $g = 9.78 \pm 1 \text{ m s}^{-2}$ .

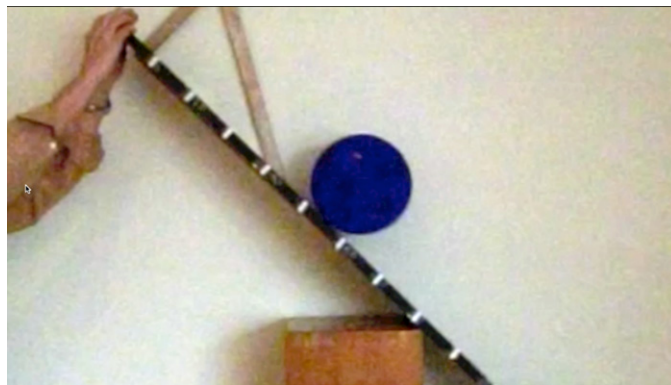
The main sources of error in our experiment were the initial position of the cylinder, which was taken in such a way that its centre of mass lay normal to the inclined plane, and the site where the cylinder actually jumps.

The initial position of the cylinder was set manually using as a guide a square fixed perpendicularly to the inclined plane at the end where the motion started (see figures 5–7). The centre of mass lay on a line from the cylinder's point of contact to the centre of the embedded rod. We estimate an error of about  $3^\circ$  in the initial position. The jumping site is difficult to determine in the video (available from [stacks.iop.org/EJP/33/1359/mmedia](https://stacks.iop.org/EJP/33/1359/mmedia)), especially in the  $45^\circ$  case, because the jump takes place before the disc has completed a whole turn and its speed is relatively small, so the jump is minute. We estimate a maximum error of about 5.0 cm in the measurements of the jumping site. Taking this into account, no error analysis was attempted; instead, we report our experimental results in the jumping site with a maximum error of  $\pm 5.0$  cm.

Figures 5–7 are photograms, taken from the videos of the jumping sites (available from [stacks.iop.org/EJP/33/1359/mmedia](https://stacks.iop.org/EJP/33/1359/mmedia)), and in table 1 we compare the theoretical solution with the experimental results.



**Figure 6.** Photogram at the jumping site for an inclined plane at  $\alpha = 30^\circ$ .



**Figure 7.** Photogram at the jumping site for an inclined plane at  $\alpha = 45^\circ$ .

**Table 1.** Comparison of the experimental results with the theoretical predictions.

Inclined plane angle $\alpha$ (deg)	15	30	45
Theoretical jumping angle $\theta_r$ (rad)	18.01	11.46	5.34
Theoretical jumping position (cm)	180.1	114.6	53.4
Experimental jumping position (cm)	$185.0 \pm 5.0$	$115.0 \pm 5.0$	$55.0 \pm 5.0$

Of special interest (to be seen in the videos) are: the  $15^\circ$  case, where the speeding up and slowing down of the cylinder are clearly seen, and the spectacular second and third jumps in the  $45^\circ$  case. The videos can be seen on YouTube <sup>1</sup>.

Along with the paper, we present an animated plot of the graphics in figures 2–4, in which we show the evolution of  $\dot{\theta}^2/g$  versus  $\cos \alpha/d \cos \theta$  with respect to the inclination angle  $\alpha$  of

<sup>1</sup> <http://www.youtube.com/watch?v=iuemR3XtSSE>.

the plane, within the interval from 0 to  $\pi/2$ . From the animation, it is clear that when  $\alpha \rightarrow 0$ , the jump would never take place, while when  $\alpha \rightarrow \pi/2$  the jumping site positions tend to  $\theta = 0$ .

#### 4. Conclusion

The first integral of the equation of motion of a cylinder whose centre of mass is not at its geometrical centre, and rolls on an inclined plane, is obtained. From this solution, the site where the cylinder should ‘jump’ is determined. An experimental setup that resembles the assumptions made to obtain the theoretical solution is furnished. The experimental results agree well with the theory. We believe that this is the first time that this problem has been theoretically and experimentally addressed.

#### Acknowledgments

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