Teaching Numerical ODE Solving using a Chaotic System

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Abstract

- A simple non-linear dynamical system with chaotic properties is used to illustrate the advantages and limitations of Runge-Kutta (RK) based ODE solving. Herein we describe the course Computer Applications in Engineering 2 (ME 323): how it fits in the ME curriculum, and course objectives. We quickly review the techniques of fixed and adaptive step fourth order RK (RK4). The definition of stability for non-linear autonomous systems is reviewed. We then present the physical system and its ODE representation. Results are shown for adaptive and fixed-step RK4 where the system stability boundary estimate visibly changes due to numerical inaccuracies.

Agenda

- Course Background
- Runge Kutta Review
  - Fixed Step
  - Adaptive Step
- Stability defined
- The System
- Results

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Course Background

- ME flowchart
- Course syllabus
- Course Schedule

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Quick Review - RK4

$$g_1 = hf(t_k, y_k)$$

$$g_2 = hf\left(t_k + \frac{1}{2}h, y_k + \frac{1}{2}g_1\right)$$

$$g_3 = hf\left(t_k + \frac{1}{2}h, y_k + \frac{1}{2}g_2\right)$$

$$g_4 = hf(t_k + h, y_k + g_3)$$

$$y_{k+1} = y_k + \frac{1}{6}g_1 + \frac{1}{3}g_2 + \frac{1}{3}g_3 + \frac{1}{6}g_4$$

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Review-Adaptive Step RK45

- Typically done by an embedded algorithm.
- RK5 is:

$$g_1 = hf(t_k, y_k)$$
$$g_2 = hf(t_k + a_2 h, y_k + b_{21} g_1)$$
$$\vdots$$
$$g_6 = hf(t_k + a_6 h, y_k + b_{61} g_1 + \cdots + b_{65} g_5)$$
$$y_{k+1} = y_k + c_1 g_1 + c_2 g_2 + c_3 g_3 + c_4 g_4 + c_5 g_5 + c_6 g_6$$

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- The embedded 4th order update is:

$$y_{k+1}^* = y_k + c_1^* g_1 + c_2^* g_2 + c_3^* g_3 + c_4^* g_4 + c_5^* g_5 + c_6^* g_6$$

- The error estimate is

$$\Delta \equiv y_{k+1} - y_{k+1}^* = \sum_{i=1}^{6} \left(c_i - c_i^*\right) g_i$$

- The desired step size is set to:

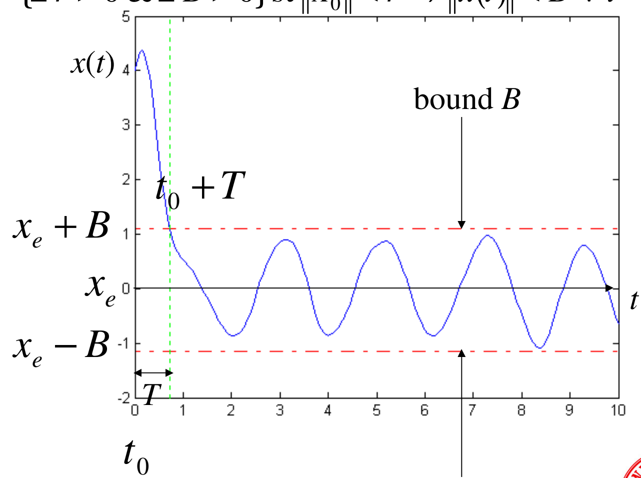
$$h_0 = h_1 \left| \frac{\Delta_0}{\Delta_1} \right|^{0.2}$$

- The coefficients used in ode45 are those in Table 2 of the Dormand-Prince paper:

a_i	b_{ij}						c_i^*	c_i
0							$\frac{35}{384}$	$\frac{5179}{57600}$
$\frac{1}{5}$	$\frac{1}{5}$						0	0
$\frac{3}{10}$	$\frac{3}{40}$	$\frac{9}{40}$					$\frac{500}{1113}$	$\frac{7571}{16695}$
$\frac{4}{5}$	$\frac{44}{45}$	$-\frac{56}{15}$	$\frac{32}{9}$				$\frac{125}{192}$	$\frac{393}{640}$
$\frac{8}{9}$	$\frac{19372}{6561}$	$-\frac{25360}{2187}$	$\frac{64448}{6561}$	$-\frac{212}{729}$			$-\frac{2187}{6784}$	$\frac{92097}{339200}$
1	$-\frac{9017}{3168}$	$-\frac{355}{33}$	$\frac{46732}{5247}$	$\frac{49}{176}$	$-\frac{5103}{18656}$		$\frac{11}{84}$	$\frac{187}{2100}$
1	$\frac{35}{384}$	0	$\frac{500}{1113}$	$\frac{125}{192}$	$-\frac{2187}{6784}$	$\frac{11}{84}$	0	$\frac{1}{40}$

Stability of Autonomous Non-Linear Systems (Lewis, Campos, Selmic)

$$\{\exists r > 0 \ \& \ \exists B > 0\} \text{ st } \|x_0\| < r \Rightarrow \|x(t)\| < B \ \forall t$$



Definition of Uniform Ultimate Boundedness

- The equilibrium point is uniformly ultimately bounded (UUB) if there exists a compact set *r* such that for every initial *x* in *r*, there exists a bound *B* and a time *T* such that *x* remains within *B* of the equilibrium point for all times greater than *T*

The System

$$-mg\sin\theta = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$$

$$\dot{r} = \text{constant (negative)}$$

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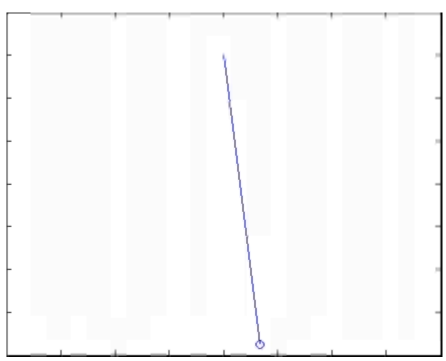
The System - Animation



- System is stable if $-\pi/2<\theta<\pi/2$ for $0.5<r<34$

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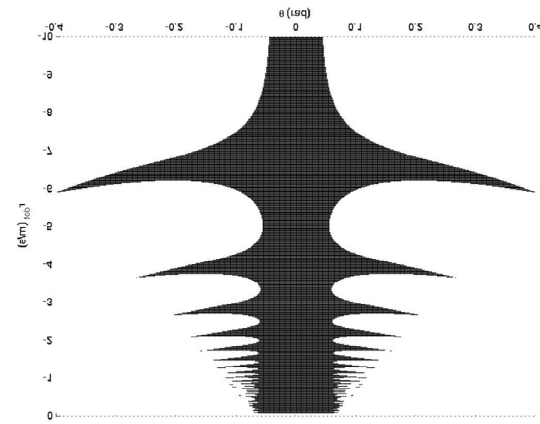
Stability boundary- precise solution:



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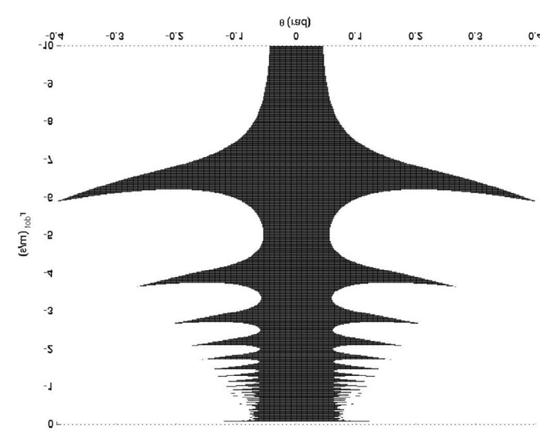
Stability Boundary: Using Fixed-Step RK4, 800 steps per pull:



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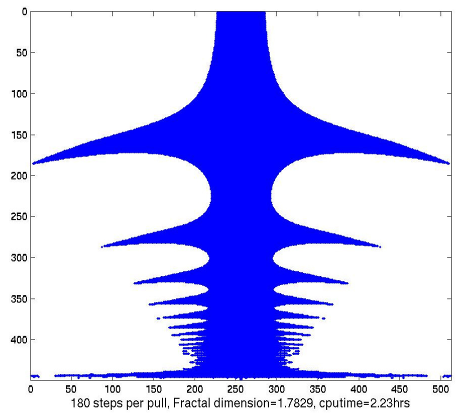
Using Fixed-Step RK4, 180 steps per pull:



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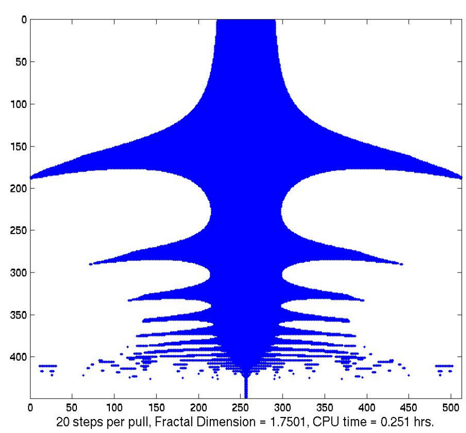
Using Fixed-Step RK4, 20 steps per pull:



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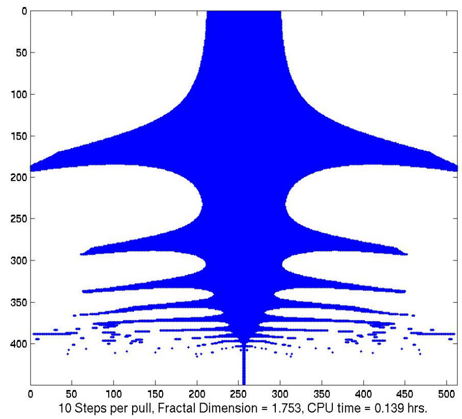
Using Fixed-Step RK4, 10 steps per pull:



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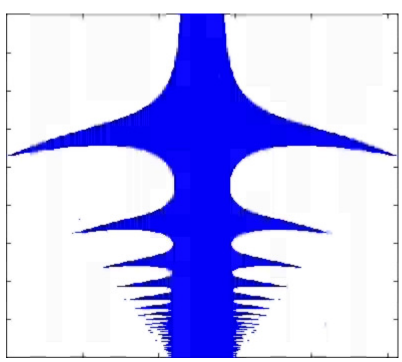
Put it all together:



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