Batch Calculation of the Residues and Their Sensitivities

Or: How to compute almost any derivative using

```
>>sum(prod([combnk(factors), dfactors.']))
```

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Outline

- Motivation - Brute force Control System Optimization
- Sylvester's Expansion
- Partial Fraction Expansion - Batch Method
- Numerical Examples
- Derivatives of the Residues
- Numerical Examples
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Motivation: Brute Force Optimization of Control Systems

- Infinite horizon optimal linear feedback controllers are determined by minimizing the cost function

$$\tilde{J} = \int_0^{\infty} \left(\mathbf{x}^T \tilde{\mathbf{Q}} \mathbf{x} + \mathbf{u}^T \tilde{\mathbf{R}} \mathbf{u}\right) \, dt$$

- Subject to the constraints defined by the system state dynamics

$$\dot{\mathbf{x}} = \tilde{\mathbf{A}}\mathbf{x} + \tilde{\mathbf{B}}\mathbf{u}$$

$$\mathbf{y} = \tilde{\mathbf{C}}\mathbf{x} + \tilde{\mathbf{D}}\mathbf{u}$$

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Motivation continued

- In the case of State Feedback, there is a direct solution from the Algebraic Ricatti Equation
- In the case of Output feedback, only iterative solutions are available, the most historic involving a pair of Lyapunov equations.
- Linear dynamics always have a closed-form solution.
- Idea: substitute the closed-loop system closed-form solution into the cost function to transform the optimization into an *unconstrained* optimization.
- The reformulated output feedback problem is termed 'brute force optimization of control systems'

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A Plethora of methods

- The closed-form solution of the state dynamics for any time is written in terms of the matrix exponential of $\mathbf{A}t$
- Thus for every method of computing the matrix exponential, there exists a corresponding brute force optimization method
- Investigations to date have used the following methods
  - Dyadic decomposition (Burchett, Costello 1997)
  - Pade Approximation (Burchett, Costello 2001)
  - Sylvester's expansion (current unpublished work)

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Sylvester's expansion

- Sylvester's expansion for systems with roots of multiplicity m_k is written as:

$$e^{\mathbf{A}t} = \sum_{k=1}^{\sigma} \sum_{l=0}^{m_k-1} t^l e^{\lambda_k t} \frac{1}{l!} (\mathbf{A} - \lambda_k \mathbf{I})^l \prod_{\substack{i=1 \\ i \neq k}}^{\sigma} (\mathbf{A} - \lambda_i \mathbf{I})^{m_i} \, n_k(\mathbf{A})$$

- n_k is the combined numerator polynomial from the partial fraction expansion of:

$$\frac{1}{\Phi(\lambda)} = \frac{n_k}{(\lambda - \lambda_k)^{m_k}} + \frac{n_{k+1}}{(\lambda - \lambda_{k+1})^{m_{k+1}}} + \ldots$$

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- Using Sylvester Expansion, the quadratic cost function can be re-written

$$\tilde{J} = \int_0^\infty \sum_{a=1}^{\sigma}\sum_{p=1}^{m_a} \mathbf{E}_{ap}^H \frac{1}{(p-1)!}\left(\mathbf{A}^H - \lambda_a^H \mathbf{I}\right)^{p-1} t^{p-1} e^{\lambda_a^H t}\mathbf{Q}$$
$$\bullet \sum_{b=1}^{\sigma}\sum_{q=1}^{n_b} t^{(q-1)} e^{\lambda_b t}\frac{1}{(q-1)!}\left(\mathbf{A} - \lambda_b \mathbf{I}\right)^{(q-1)} \mathbf{E}_{bq}\, dt$$

- Where

$$\mathbf{E}_k = \frac{\Phi(\mathbf{A}) n_k(\mathbf{A})}{(\lambda - \lambda_k)^{m_k}}$$

- $\Phi(\mathbf{A})$ is the minimum polynomial of $\mathbf{A}$

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- The cost function in this form can be integrated closed-form yielding (this requires integration by parts and invoking mathematical induction)

$$\tilde{J} = \sum_{a=1}^{\sigma}\sum_{b=1}^{\sigma}\sum_{p=1}^{m_a}\sum_{q=1}^{n_b} \frac{(-1)^{p+q-1}(p+q-2)!}{(\lambda_a^{\dagger} + \lambda_b)^{p+q-1}(p-1)!(q-1)!}$$
$$\bullet \mathbf{E}_{ap}^H \left(\mathbf{A}^H - \lambda_a^H \mathbf{I}\right)^{(p-1)} \mathbf{Q} \left(\mathbf{A} - \lambda_b \mathbf{I}\right)^{(q-1)} \mathbf{E}_{bq}$$

- We would like to invoke gradient based methods with analytic derivatives if possible
- This requires computing the derivatives of all factors in the equation above (including eigenvalues and residues).

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Partial Fraction Expansion

- The partial fraction expansion of a system transfer function

$$F(s) = \frac{B(s)}{A(s)}$$

$$\frac{B(s)}{A(s)} = \frac{K\,(s+z_1)\,(s+z_2)\cdots(s+z_m)}{(s+p_1)\,(s+p_2)\cdots(s+p_n)}, \ \texttt{for}\ m < n$$

- Can be written

$$\frac{B(s)}{A(s)} = \frac{a_1}{(s+p_1)} + \frac{a_2}{(s+p_2)} + \cdots + \frac{a_n}{(s+p_n)}$$

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- Adding the RHS of the previous eqn by finding a common denominator then equating powers of *s* in the numerators right and LHS of the previous eqn, a set of *n* linear equations emerges. This set of equations can be written in a matrix form

$$\mathbf{\Xi a = H}$$

- **H** is a vector of coefficients from the convolved form of the original numerator polynomial

$$B(s) = \eta_1 s^{n-1} + \eta_2 s^{n-2} + \cdots + \eta_n$$

$$\mathbf{H} = [\eta_1\ \eta_2\ \cdots\ \eta_n]^T$$

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- $\mathbf{\Xi}$ has a distinct pattern in terms of row i and column j.

$$\Xi_{i,j} = \sum_{\substack{k,l=\binom{n-1}{i-1} \\ k,l\neq j}} (-p_k)(-p_l)\cdots$$

- In Matlab code, that is

```
»Xi(i,j)=sum(prod(combnk(-poles([1:j-1, j+1:n]),i-1)));
```

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- For the distinct poles case:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \sum_{j=2}^{m-1} -p_j & \sum_{\substack{j=1 \\ j\neq 2}}^{m-1} -p_j & \cdots & \sum_{j=1}^{m-2} -p_j \\ \sum_{\substack{jk=\binom{m-1}{2} \\ jk\neq 1}} (-p_j)(-p_k) & \sum_{\substack{jk=\binom{m-1}{2} \\ jk\neq 2}} (-p_j)(-p_k) & \cdots & -\sum_{\substack{jk=\binom{m-1}{2} \\ jk\neq m-1}} (-p_j)(-p_k) \\ \vdots & \vdots & \vdots & \vdots \\ \prod_{j=2}^{m-1} -p_j & \prod_{\substack{j=1 \\ j\neq 2}}^{m-1} -p_j & \cdots & \prod_{j=1}^{m-2} -p_j \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ \vdots \\ n_{m-1} \end{Bmatrix} = \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{m-1} \end{Bmatrix}$$

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- For systems with repeated poles, the PFE is:

$$\frac{B(s)}{A(s)} = \frac{a_1}{(s+p_1)^{m_k}} + \frac{a_2}{(s+p_1)^{m_k-1}} + \cdots + \frac{a_{m_k}}{(s+p_1)} + \cdots + \frac{a_{m_k+1}}{(s+p_2)} + \cdots \frac{a_n}{(s+p_n)} \quad (5)$$

- By repeating the algebraic process outlined in above, we discover that the resulting system of equations **Ξa=H** has the same form as Eq. 4, except that for the terms involving repeated poles, the corresponding column of the **Ξ** matrix can be built from the bottom up using Eq. 4, pretending that the system is lacking m_k-j+1 occurrences of the repeated pole.

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- Or more specifically:

$$\begin{bmatrix} 0 & 0 & \cdots & 1 & \\ \vdots & \vdots & \cdots & \sum_{k=1}^{n-1}(-p_k) - (m_k - 1)p_1 & \\ 0 & 1 & \cdots & \sum_{\substack{k,l=\binom{n-1}{2} \\ k,l\neq n}} (-p_k)(-p_l) & \\ 1 & \sum_{\substack{k=1 \\ k=\neq 2}}^{n} -p_k - p_1 & \cdots & \sum_{\substack{k,l,m=\binom{n-1}{3} \\ k,l\neq n}} (-p_k)(-p_l)(-p_m) & \\ \sum_{k=2}^{n} -p_k & \sum_{\substack{k,l=\binom{n-m_k+1}{2} \\ k,l\neq 2}} (-p_k)(-p_l) & \cdots & \vdots & \cdots \\ \sum_{\substack{k,l=\binom{n-1}{2} \\ k,l\neq 1}} (-p_k)(-p_l) & \sum_{\substack{k,l,m=\binom{n-m_k+1}{3} \\ k,l\neq 1}} (-p_k)(-p_l)(-p_m) & \cdots & \vdots & \\ \vdots & \vdots & & \vdots & \\ \prod_{k=2}^{n}(-p_k) & p_1 \prod_{k=2}^{n}(-p_k) & \cdots & p_1^{m_k-1} \prod_{k=2}^{n}(-p_k) & \end{bmatrix} \quad (11)$$

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Numerical Examples

- First case: distinct poles:

$$\frac{0.762s^3 + 0.457s^2 + 0.019s + 0.821}{s^4 + 0.243s^3 + 0.639s^2 + 0.512s + 0.938}$$

- The PFE is:

$$\begin{aligned} &= \frac{a_1}{(s - 0.515 - 0.911j)} + \frac{a_2}{(s - 0.515 + 0.911j)} \\ &+ \frac{a_3}{(s + 0.637 - 0.672j)} + \frac{a_4}{(s + 0.637 + 0.672j)} \end{aligned}$$

- The matrix equation is

$$[\xi_1 \quad \xi_1^\dagger \quad \xi_2 \quad \xi_2^\dagger]\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} = \begin{Bmatrix} 0.762 \\ 0.457 \\ 0.019 \\ 0.821 \end{Bmatrix}$$

- Where

$$\xi_1 = [1 \quad .758 + .911j \quad .2 + 1.16j \quad -.441 + .78j]^T,$$

$$\xi_2 = [1 \quad -.394 + .672j \quad -.439 - .692j \quad .697 + .735j]^T$$

- The residues are $a_{1,2}$=0.107±0.063j, $a_{3,4}$=0.274±0.296j

- Second case: Repeated poles

$$\frac{1}{s^4+4s^3+8s^2+8s+4}$$

- The PFE is:

$$=\frac{a_1}{(s+1+j)^2}+\frac{a_2}{(s+1-j)^2}+\frac{a_3}{(s+1+j)}+\frac{a_4}{(s+1-j)}$$

- The matrix equation is

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 3-j & 3+j \\ 2-2j & 2+2j & -4-2j & -4+2j \\ -2j & 2j & 2-2j & 2+2j \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix}$$

- The resdiues are $a_{1,2}$=-0.25, $a_{3,4}$±0.25j

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Derivatives of the Residues

- Since we have constructed the residue calculation as a linear system

$$\mathbf{a}=\mathbf{\Xi}^{-1}\mathbf{H}$$

- The derivatives are found by application of the product rule:

$$\frac{\partial \mathbf{a}}{\partial \mathbf{K}}=\frac{\partial \mathbf{\Xi}^{-1}}{\partial \mathbf{K}}\mathbf{H}+\mathbf{\Xi}^{-1}\frac{\partial \mathbf{H}}{\partial \mathbf{K}}$$

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- Then using the matrix inversion fact

$$\frac{\partial \mathbf{\Xi}^{-1}}{\partial \mathbf{K}} = -\mathbf{\Xi}^{-1}\frac{\partial \mathbf{\Xi}}{\partial \mathbf{K}}\mathbf{\Xi}^{-1}$$

- We obtain

$$\frac{\partial \mathbf{a}}{\partial \mathbf{K}} = -\mathbf{\Xi}^{-1}\frac{\partial \mathbf{\Xi}}{\partial \mathbf{K}}\mathbf{\Xi}^{-1}\mathbf{H} + \mathbf{\Xi}^{-1}\frac{\partial \mathbf{H}}{\partial \mathbf{K}}$$

- Which is good news indeed--we do not require the inverse of the matrix derivative

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- For the distinct poles case, the matrix derivative follows from the original pattern

$$\frac{\partial \Xi_{i,j}}{\partial K} = \sum_{\substack{m=1 \\ m \neq j}}^{n} \left(-\frac{\partial p_m}{\partial K} \right) \prod_{\substack{k=\binom{n-2}{i-2} \\ k \neq m}} (-p_k)$$

- In Matlab code, if

```
»rows = combnk(-poles([1:j-1, j+1:n]),k);
»column = fliplr(-dpoles(1:j-1, j+1:n));
```

- Then

```
»dXi = sum(prod(combnk(rows(m,:),k-1), column(:),2));
```

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- More explicitly:

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ \sum_{j=2}^{n-1} -\frac{\partial p_j}{\partial K} & \sum_{\substack{j=1 \\ j\neq 2}}^{n-1} -\frac{\partial p_j}{\partial K} & \cdots & \sum_{j=2}^{n} -\frac{\partial p_j}{\partial K} \\ \sum_{\substack{j,k=P_2^{n-1} \\ j,k\neq 1}} \left(-\frac{\partial p_j}{\partial K}\right)(-p_k) & \sum_{\substack{j,k=P_2^{n-1} \\ j,k\neq 2}} \left(-\frac{\partial p_j}{\partial K}\right)(-p_k) & \cdots & \sum_{\substack{j,k=P_2^{n-1} \\ j,k\neq n}} \left(-\frac{\partial p_j}{\partial K}\right)(-p_k) \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{\substack{m=1 \\ m\neq 1}}^{n} \left(-\frac{\partial p_m}{\partial K}\right) \prod_{\substack{k=\binom{n-2}{i-2} \\ k\neq m}} (-p_k) & \sum_{\substack{m=1 \\ m\neq 2}}^{n} \left(-\frac{\partial p_m}{\partial K}\right) \prod_{\substack{k=\binom{n-2}{i-2} \\ k\neq m}} (-p_k) & \cdots & \sum_{\substack{m=1 \\ m\neq n}}^{n} \left(-\frac{\partial p_m}{\partial K}\right) \prod_{\substack{k=\binom{n-2}{i-2} \\ k\neq m}} (-p_k) \end{bmatrix}$$

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Numerical Example

- Considering the Frequency domain function

$$\frac{.406s^4 + .936s^3 + .917s^2 + 0.41s + 0.894}{s^5 + 1.38s^4 + 1.78s^3 + 2.073s^2 + 1.66s + .396}.$$

- Assuming the pole derivatives are known as

$$\left\{ \begin{array}{c} \partial p_{1,2}/\partial K \\ \partial p_{3,4}/\partial K \\ \partial p_5/\partial K \end{array} \right\} = \left\{ \begin{array}{c} .0765 \mp 0.139i \\ .657 \mp .108i \\ -1.466 \end{array} \right\}$$

- The derivative matrix is

$$\frac{\partial \mathbf{\Xi}}{\partial \mathbf{K}} = [\ \xi_1 \quad \xi_1^{\dagger} \quad \xi_2 \quad \xi_2^{\dagger} \quad \xi_3\ ]$$

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- Where

$$\xi_1 = \begin{bmatrix} 0 \\ 0.0765 - 0.1386i \\ 0.4528 - 0.0884i \\ 0.5694 + 0.5150i \\ -0.2248 + 0.8838i \end{bmatrix} \quad \xi_2 = \begin{bmatrix} 0 \\ 0.6566 - 0.1078i \\ 0.0307 + 0.695i \\ 0.2897 - 0.662i \\ 1.0586 + 0.8917i \end{bmatrix}$$

$$\xi_3 = \begin{bmatrix} 0 \\ -1.4662 \\ -0.9666 \\ -1.7181 \\ -1.6674 \end{bmatrix}$$

- The derivative of the **H** matrix is taken to be

$$\frac{\partial \mathbf{H}}{\partial K} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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- The resulting residue derivatives are:

$$\left\{ \begin{array}{c} \frac{\partial a_{1,2}}{\partial \mathbf{K}} \\ \frac{\partial a_{3,4}}{\partial \mathbf{K}} \\ \frac{\partial a_5}{\partial \mathbf{K}} \end{array} \right\} = \left\{ \begin{array}{c} -0.1077 \pm 0.1635i \\ -2.5461 \mp 0.9141i \\ 5.3075 \end{array} \right\}$$

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Repeated Poles Example

- Consider again the system

$$\frac{1}{s^4+4s^3+8s^2+8s+4}$$

- With pole derivatives

$$\left\{\begin{array}{c}\partial p_{1,2}/\partial K\\ \partial p_{3,4}/\partial K\end{array}\right\}=\left\{\begin{array}{c}71.96\pm 173.59i\\ -71.96\mp 173.84i\end{array}\right\}$$

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- This leads to an enigma
  - when constructing the columns of the Ξ matrix corresponding to the repeated pole, we ignored one occurrence of the repeated pole
  - When constructing the derivative of the Ξ matirx, which pole derivative to I ignore?
- I can show what the numbers should be, from finite differencing, but I cannot determine the correct pattern...

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- The derivative of the Ξ matrix is

$$\frac{\partial \Xi}{\partial \mathbf{K}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & i/8 & 0 & -i/8 \\ i/4 & 0 & -i/4 & -1/4 - i/4 \\ 1/2 & 1/4 + i/2 & 1/4 - i/4 & -1/4 - i/2 \end{bmatrix}$$

- The residue sensitivities are

$$\left\{ \begin{array}{c} \frac{\partial a_1}{\partial \mathbf{K}} \\ \frac{\partial a_2}{\partial \mathbf{K}} \\ \frac{\partial a_3}{\partial \mathbf{K}} \\ \frac{\partial a_4}{\partial \mathbf{K}} \end{array} \right\} = \left\{ \begin{array}{c} -3.7875 - 39.7354i \\ -39.7042 + 0.125i \\ 3.6625 - 39.6729i \\ 39.7042 - 0.125i \end{array} \right\}$$

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Conclusions

- Batch calculation seems to be a viable method for finding derivatives of the residues
- The important case of systems with repeated poles needs further investigation
- Help ?!

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