Toward Finite Horizon Optimal Control of Indirect Fire Munitions

Bradley T. Burchett and Austin L. Nash,
Rose-Hulman Institute of Technology

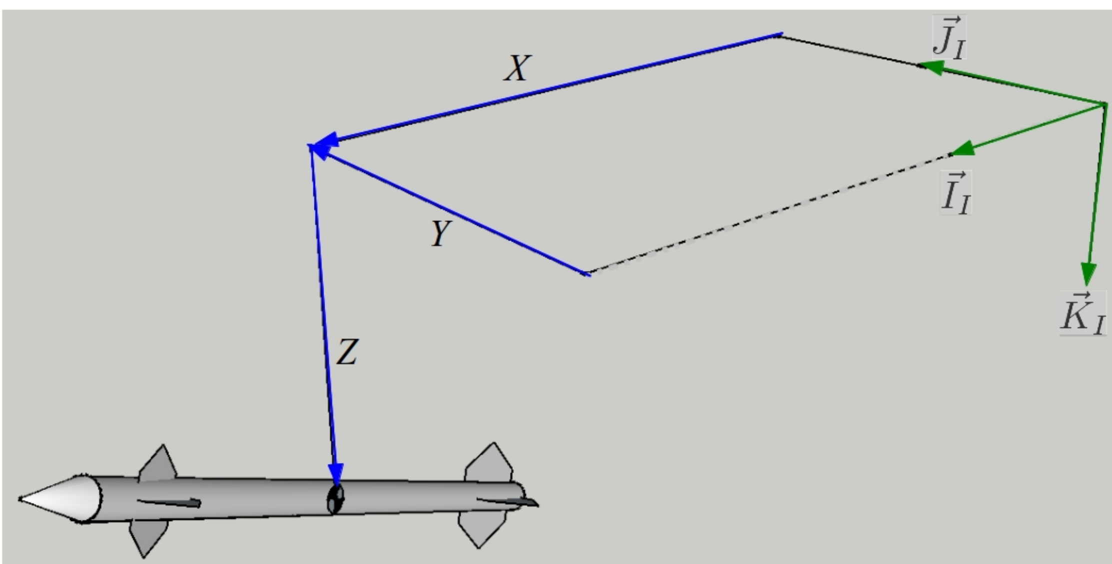
Outline

- Motivation
- Projectile coordinate frames
- Non-linear plant model
- Nine (9) state linear model for control
- LTI Finite Horizon Linear Optimal Regulator
- Time Varying Piecewise Linear Optimal Regulator
- Non-Linear Implementation
- Indirect Fire Control Strategy
- Conclusions

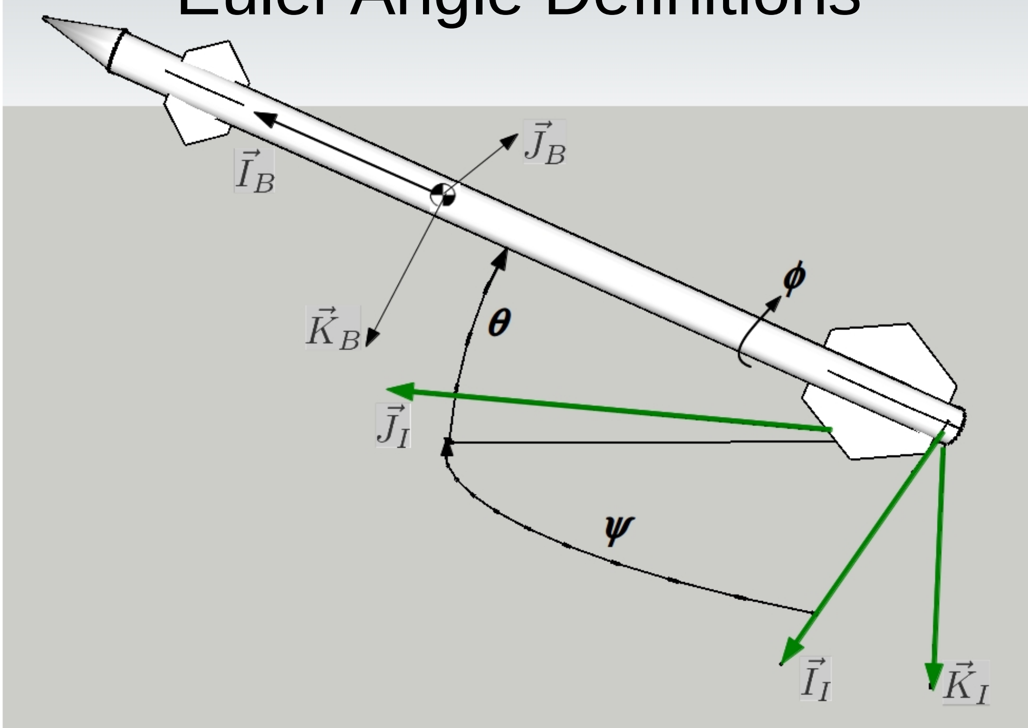
Motivation

- Recent contributions in surface-to-surface tactical missile control focus on model predictive control (MPC)
- Previous efforts predicted the impact point both with and without control
- State of the art relies on converting the plant model to discrete time and providing a reference trajectory
- In this work, the ***need for a reference trajectory is eliminated*** by
  - removing the state penalty term from the cost function
  - treating gravity as an uncontrollable mode

Projectile Coordinate Frames



Euler Angle Definitions



Non-linear plant model: position states

$$\begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{Bmatrix} = \begin{bmatrix} c_\theta c_\psi & s_\phi s_\theta s_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

$$\begin{Bmatrix} \psi \\ \dot{\theta} \\ \dot{\phi} \end{Bmatrix} = \begin{bmatrix} 1 & s_\phi t_\theta & c_\phi t_\theta \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi / c_\theta & c_\phi / c_\theta \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

The World's Forum for Aerospace Leadership

Nonlinear plant model velocity states

$$\begin{Bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{Bmatrix} = \frac{1}{m}\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} - \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{Bmatrix} = [I]^{-1} \left[\begin{Bmatrix} L \\ M \\ N \end{Bmatrix} - \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} [I] \begin{Bmatrix} p \\ q \\ r \end{Bmatrix} \right]$$

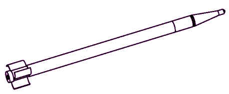
The World's Forum for Aerospace Leadership

Conventional Projectile linear theory assumptions

- Change of variable from SL velocity *u* to total velocity *V*
- Change of independent variable *t* to arclength *s*
- Euler yaw and pitch angles are small
- Aerodynamic angles of attack are small
- Projectile is mass balanced (axially symmetric)
- Projectile is aerodynamically axially symmetric
- Flat fire trajectory assumed
- *V* and ϕ are large, products of other quantities and their derivatives are negligible

Nine State Linear Model

$$\begin{Bmatrix} \dot{\xi} \\ \dot{\eta} \\ \ddot{w} \end{Bmatrix} = \begin{bmatrix} \mathbf{\Phi} & \mathbf{\Gamma} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Xi} & \mathbf{\Lambda} \\ \mathbf{0} & \mathbf{0} & 0 \end{bmatrix} \begin{Bmatrix} \xi \\ \eta \\ \dot{w} \end{Bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{b} \\ \mathbf{0} \end{bmatrix} \begin{Bmatrix} C_{Z0} \\ C_{Y0} \end{Bmatrix}$$



$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\xi = \begin{bmatrix} y & z & \theta & \psi \end{bmatrix}^T, \ \eta = \begin{bmatrix} v & w & q & r \end{bmatrix}^T, \ \mathbf{\Lambda} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T, \ \mathbf{\Gamma} = \frac{D}{V}\mathbf{I}$$

$$\mathbf{\Phi} = \begin{bmatrix} 0 & 0 & 0 & D \\ 0 & 0 & -D & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \mathbf{\Xi} = \begin{bmatrix} -\mathbf{\Xi}_1 & 0 & 0 & -D \\ 0 & -\mathbf{\Xi}_1 & D & 0 \\ \mathbf{\Xi}_2 & \mathbf{\Xi}_3 & \mathbf{\Xi}_4 & -\mathbf{\Xi}_5 \\ -\mathbf{\Xi}_3 & \mathbf{\Xi}_2 & \mathbf{\Xi}_5 & \mathbf{\Xi}_4 \end{bmatrix}$$

LTI Finite Horizon Optimal Control

- System matrices treated as constants such that **A**(*p*,*V*) = **A**, etc.:
- Form the finite horizon cost function:

$$J = \frac{1}{2}\mathbf{x}^T(s_t)\mathbf{P}\mathbf{x}(s_t) + \int_{s_i}^{s_t} \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\, ds$$

- Since **Q** ≥ 0, choose **Q** = 0, then define the Hamiltonian:

$$\tilde{H} = \frac{1}{2}\mathbf{u}^T\mathbf{R}\mathbf{u} + \lambda^T(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u})$$

LTI Finite Horizon Optimal Control, Cont'd

- Taking variations yields the conditions for optimality:

$$\dot{\lambda}^T = -\frac{\partial \tilde{H}}{\partial \mathbf{x}} = -\lambda^T \mathbf{A}$$

$$\frac{\partial \tilde{H}}{\partial \mathbf{u}} = \mathbf{0} \rightarrow \mathbf{0} = \mathbf{u}^T \mathbf{R} + \lambda^T \mathbf{B}$$

- Thus the control law is

$$\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^T \lambda$$

LTI Finite Horizon Optimal Control, Cont'd

- State and co-state ODEs are collected into a single matrix equation:

$$\begin{Bmatrix} \dot{\mathbf{x}} \\ \dot{\lambda} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T \\ \mathbf{0} & -\mathbf{A}^T \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \lambda \end{Bmatrix}$$

- Which can be solved using state transition matrices such that:

$$\begin{Bmatrix} \mathbf{x}(s) \\ \lambda(s) \end{Bmatrix} = \begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{x}(s_t) \\ \mathbf{P}\mathbf{x}(s_t) \end{Bmatrix}$$

LTI Optimal Control cont'd

- Such that the current co-state is given by:

$$\lambda(s) = [\mathbf{\Sigma}_{21} + \mathbf{\Sigma}_{22}\mathbf{P}] \bullet [\mathbf{\Sigma}_{11} + \mathbf{\Sigma}_{12}\mathbf{P}]^{-1}\mathbf{x}(s)$$

- And the transition matrices are given by the full 18 x 18 matrix exponential:

$$\begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{bmatrix} = \exp \begin{bmatrix} \mathbf{A}\sigma & -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\sigma \\ \mathbf{0} & -\mathbf{A}^T\sigma \end{bmatrix}$$

Time Varying Piecewise Linear Optimal Control

- System matrices become time varying such that **A**(*s*) = **A**(*p*(*s*),*V*(*s*)), etc.:

$$\mathbf{u}(s) = -\mathbf{R}^{-1}\mathbf{B}^T(s)\mathbf{N}(s)\mathbf{x}(s)$$

- **N**(*s*) is the solution to the time varying Riccati eqn:

$$\dot{\mathbf{N}}(s) = -\mathbf{N}(s)\mathbf{A}(s) - \mathbf{A}^T(s)\mathbf{N}(s) + \mathbf{N}(s)\mathbf{B}(s)\mathbf{R}^{-1}\mathbf{B}^T(s)\mathbf{N}(s) - \mathbf{Q}$$

Decompose the Riccati Eqn. Into two DEs:

$$\begin{Bmatrix} \dot{\mathbf{W}}(s) \\ \dot{\mathbf{Y}}(s) \end{Bmatrix} = \begin{bmatrix} \mathbf{A}(s) & -\mathbf{B}(s)\mathbf{R}^{-1}\mathbf{B}^T(s) \\ 0 & -\mathbf{A}^T(s) \end{bmatrix} \begin{Bmatrix} \mathbf{W}(s) \\ \mathbf{Y}(s) \end{Bmatrix}$$

- Or: $\dot{\mathbf{Z}}(s) = \mathbf{F}(s)\mathbf{Z}(s)$
- And the solution is:

$$\mathbf{N}(s) = \mathbf{Y}(s)\mathbf{W}(s)^{-1}$$

LPTV Control Continued

- Must predict *p, V* from current state to target (however roughly).
- From previous work (Burchett, 2001)

$$p(s+h) = p(s)\Lambda + \frac{2V(s)C_{LDD}}{DC_{LP}} \exp\left(-\frac{\rho SDC_{X0}}{2m}h\right)(\Lambda - 1)$$

$$\Lambda = \exp\left(\frac{\rho SD^3 C_{LP}}{4I_{xx}}h\right)$$

$$V(s+h) = V(s)\exp\left(-\frac{\rho SDC_{X0}}{2m}h\right)$$

LPTV Control Continued

- Solve the Riccati set recursively backward in time from the target to current state

$$\mathbf{Z}_{n_s} = \left(\mathbf{I} + \frac{h}{2}\mathbf{F}_{n_s}\right)^{-1} \mathbf{Z}(s_t)$$

$$\mathbf{Z}_k = \left(\mathbf{I} + \frac{h}{2}\mathbf{F}_k\right)^{-1} \left(\mathbf{I} - \frac{h}{2}\mathbf{F}_{k+1}\right) \mathbf{Z}_{k+1},\ k = 0, 1, \ldots, n_s - 1$$

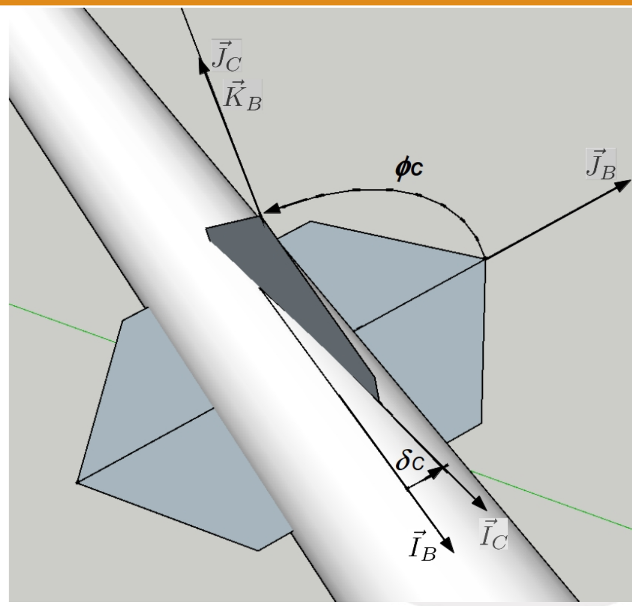
Dou and Dou, 2012

LPTV Control Summary

- Calculate the distance to target
- Divide this distance into n_s equal segments
- Recursively estimate the total *V* and *p* updating aero coefficients at each segment
- Build the corresponding Hamiltonian matrix for each segment
- Integrate backwards in time.
- Use $\mathbf{Z}_1$ to compute Riccati solution at current state
- Compute control in no roll frame

AIAA
The World's Forum for Aerospace Leadership

Non-linear Implementation, frames



AIAA
The World's Forum for Aerospace Leadership

- Coordinate rotation to canard frame

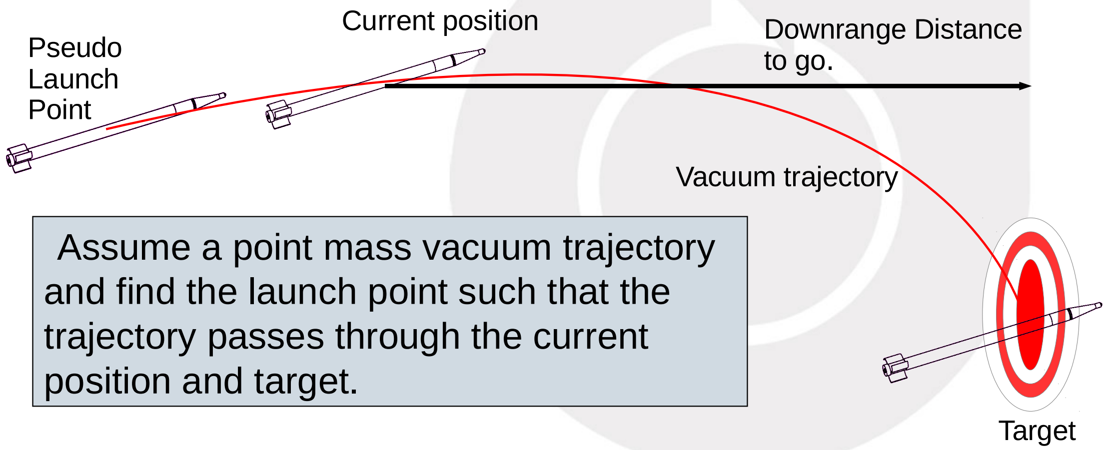
$$\left\{ \begin{array}{c} \delta_Z \\ \delta_Y \end{array} \right\}_R = \left[\begin{array}{cc} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{array} \right] \left\{ \begin{array}{c} \delta_Z \\ \delta_Y \end{array} \right\}_{NR}$$

- Reverse table lookup for canard deflection

$$\delta_{can}[\mathtt{rad}] = C_{Y0}/C_{L\alpha}$$

Non-linear implementation enhancements

Arc length to go is estimated by first finding the pseudo launch point.



Assume a point mass vacuum trajectory and find the launch point such that the trajectory passes through the current position and target.

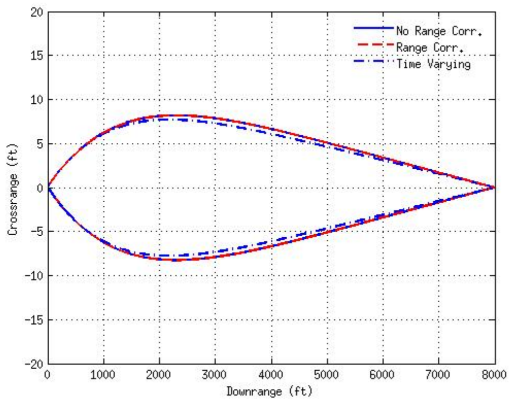
Arclength to Go

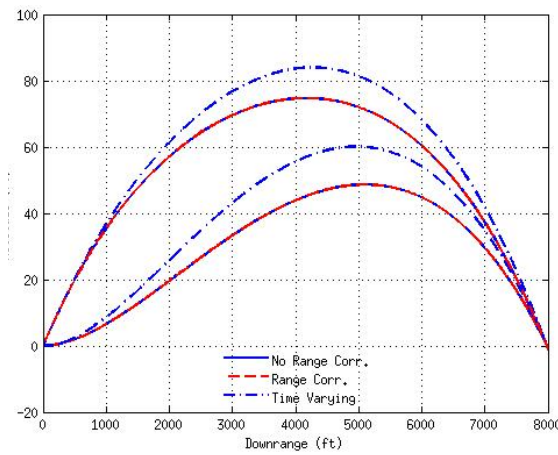
$$z(x) = z_0 + V_{0z}\frac{x}{V_{0x}} + \frac{a}{2}\left(\frac{x}{V_{0x}}\right)^2$$

Next, integrate from current downrange position to target.

$$\int ds = \int_x^{x_t} \sqrt{\left(\frac{V_{0z}}{V_{0x}} + \frac{a}{V_{0x}^2}x\right)^2 + 1}\ dx$$

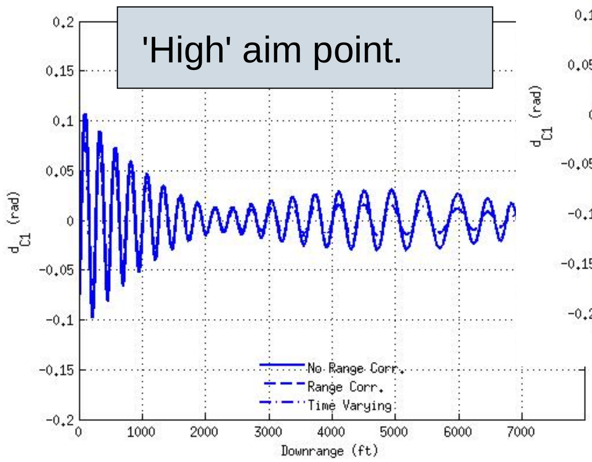
Comparison, typical Controlled trajectories from the three control strategies:



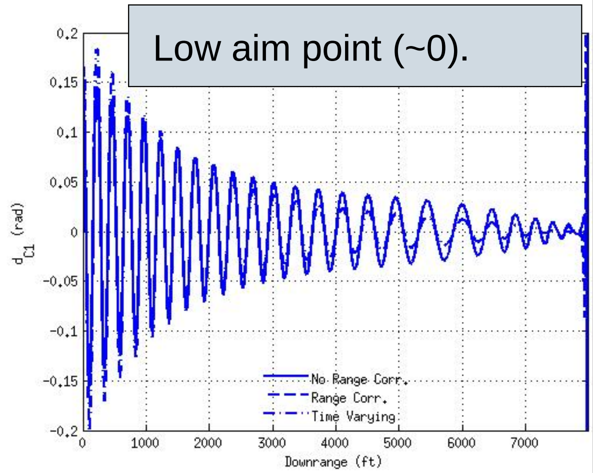


Control Effort is within reasonable bounds

'High' aim point.

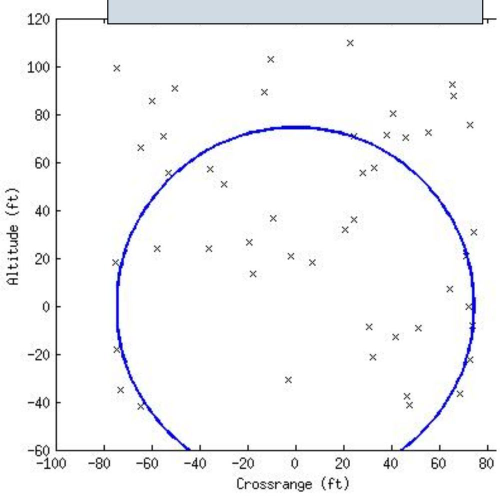


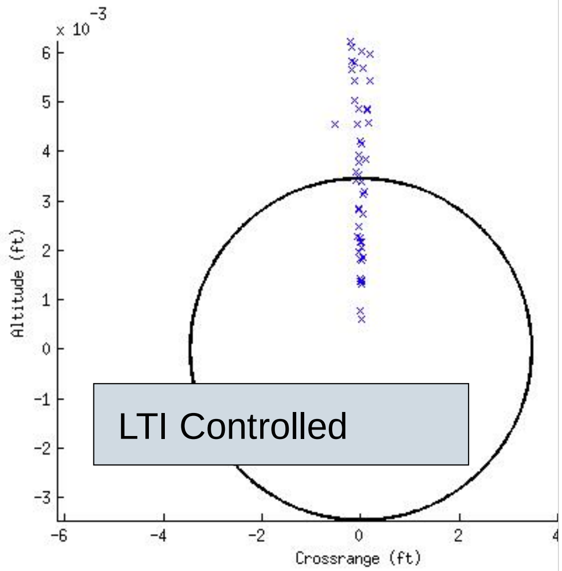
Low aim point (~0).



Dispersions, Uncontrolled, LTI controller

Uncontrolled

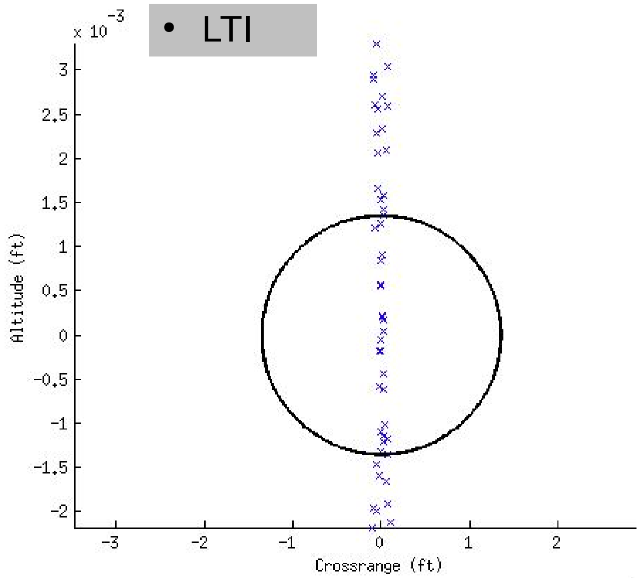




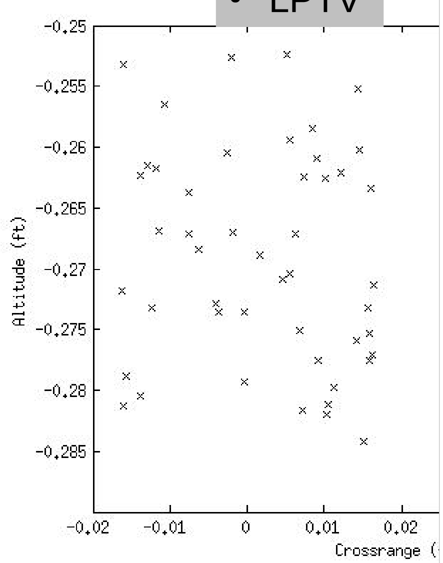
LTI Controlled

Dispersions, with Range correction, LTI, LPTV

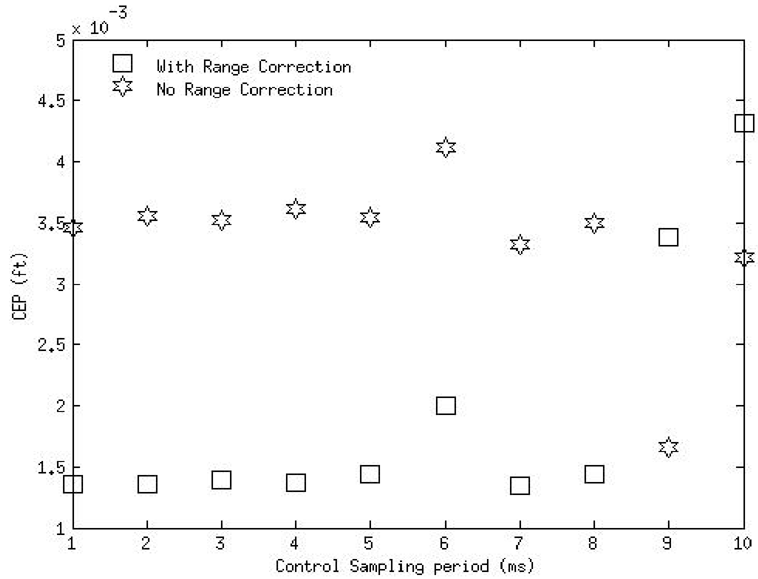
- LTI



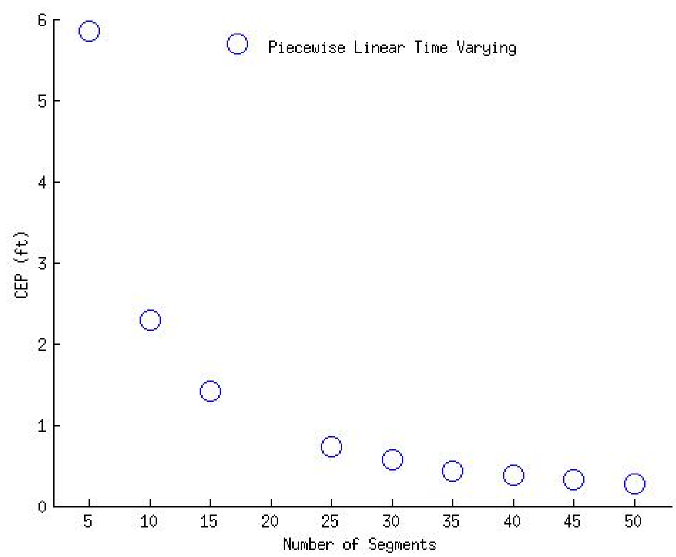
- LPTV



LTI Trade Study, Control Sampling period



LPTV Trade Study, Number of Segments



Modifications for Indirect Fire

- Form the Hamiltonian from Time Varying Modified Linear Theory Equations
- Predict Time varying parameters *p*, *V*, cos(θ), and sin(θ) from vacuum point mass trajectory
- Base the vacuum trajectory on origin launch point and solve for both initial *x* and *y* velocities

Modified Linear Theory Model

$$\begin{Bmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{w} \end{Bmatrix} = \begin{bmatrix} \Phi & \Gamma & \Sigma \\ 0 & \Xi & \Lambda \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \xi \\ \eta \\ \dot{w} \end{Bmatrix} + \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} \begin{Bmatrix} C_{Z0} \\ C_{Y0} \end{Bmatrix}$$

$$\Phi = DC_{\theta} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Costello and Hainz, 2005

Modified arrays

$$\Gamma = \frac{D}{V}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{c_\theta} \end{bmatrix}$$

$$\Sigma = D\, s_\theta \begin{bmatrix} 0 & -1 & 0 & 0 \end{bmatrix}$$

Modified arrays, cont'd

$$\Lambda = \frac{Dg}{V} c_\theta \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

- State vector is modified to

$$\xi = \begin{bmatrix} y & z & \delta_\theta & \psi \end{bmatrix}^T$$

$$\eta = \begin{bmatrix} v & w & q & r \end{bmatrix}^T$$

A new LTV paradigm?

- In essence, what we have done is taken the Taylor series approximation

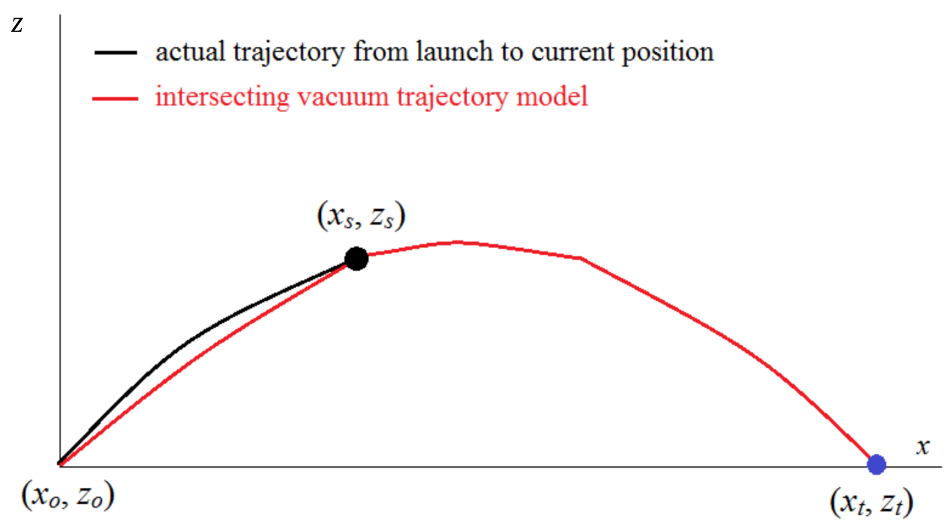
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \mathbf{f}(\bar{\mathbf{x}}) + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}},\bar{\mathbf{u}}} \delta\mathbf{x} + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\bar{\mathbf{x}},\bar{\mathbf{u}}} \delta\mathbf{u}$$

- And reformulated it as a **stabilizable, time varying linear system**

$$\left\{ \begin{array}{c} \dot{\mathbf{x}} \\ \dot{1} \end{array} \right\} = \left[\begin{array}{cc} \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}},\bar{\mathbf{u}}} & \mathbf{f}(\bar{\mathbf{x}}) \\ \mathbf{0} & 0 \end{array} \right] \left\{ \begin{array}{c} \delta\mathbf{x} \\ 1 \end{array} \right\} + \left[\begin{array}{c} \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\bar{\mathbf{x}},\bar{\mathbf{u}}} \\ 0 \end{array} \right] \delta\mathbf{u}$$

- TBD: what other problems fall into this class?

Point Mass Vacuum Trajectory Geometry



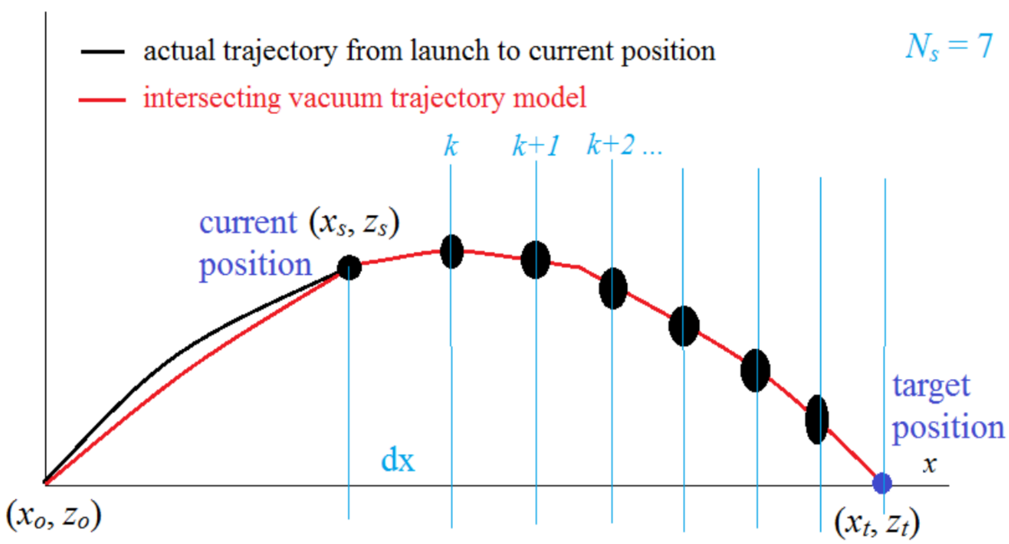
Vacuum traj. model

- This time write two instances of

$$z(x) = z_0 + V_{0z}\frac{x}{V_{0x}} + \frac{a}{2}\left(\frac{x}{V_{0x}}\right)^2$$

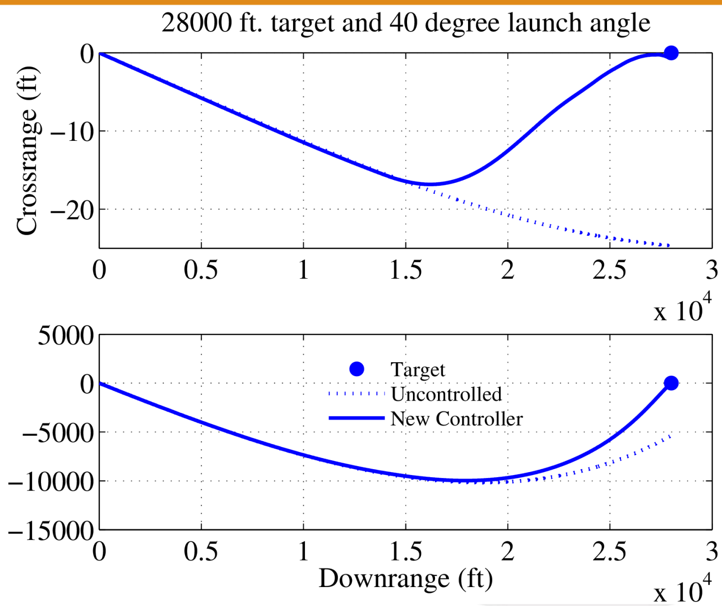
- At current state z_k and x_k and target state (x_k,0) treating both velocities as unknowns

Divide remaining trajectory into segments

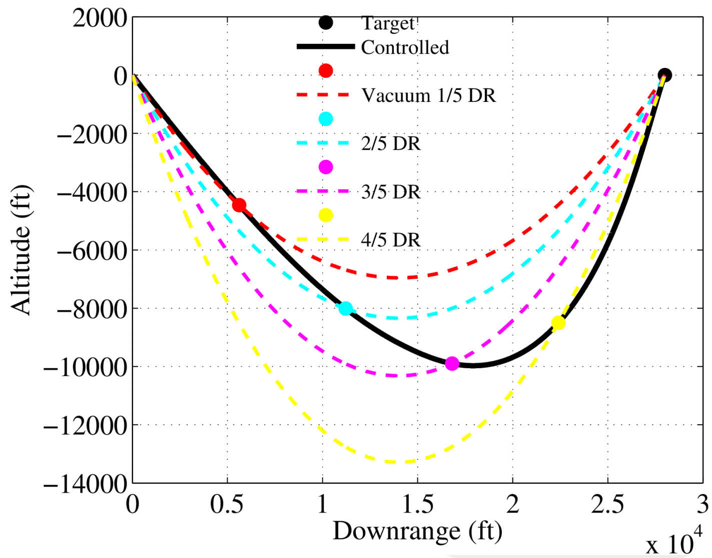


- Predict p, V, $\cos(\theta)$, $\sin(\theta)$
- Set up time varying Hamiltonian at each segment
- Solve the time varying Riccati equation backwards in time from the target as shown previously
- Convert normal force coeffs into deflections and convert to missile body frame as before

Typical performance for indirect fire



Controlled approaches vacuum near the target



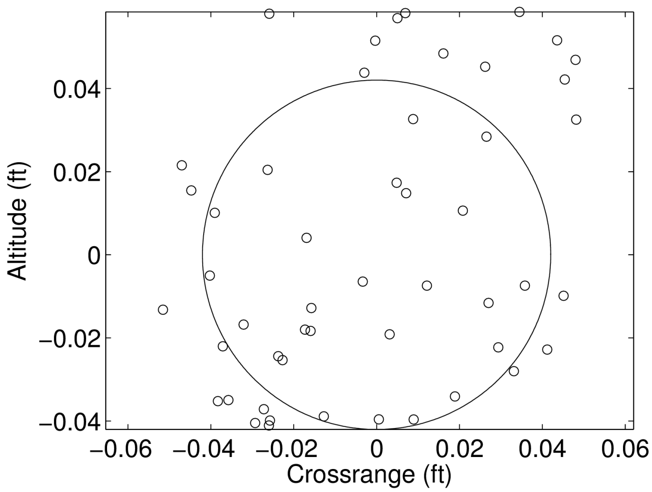
Conclusions

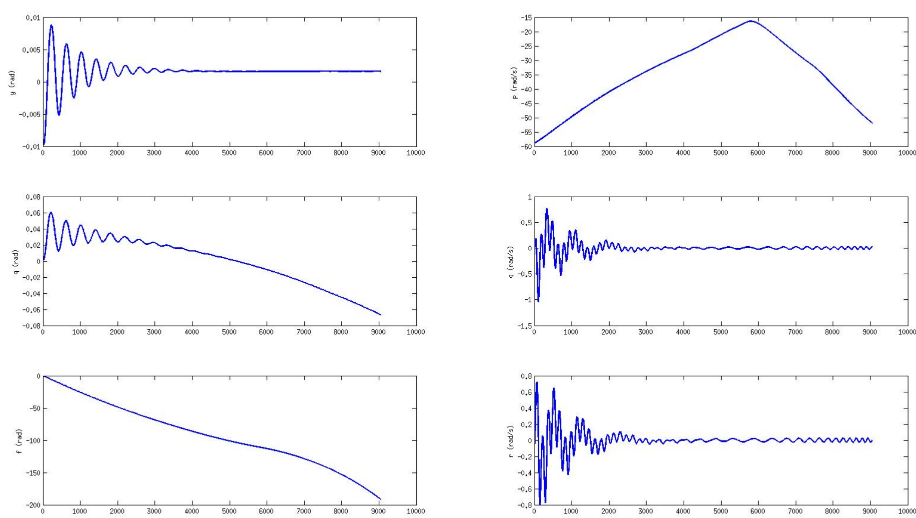
- Three control strategies for direct fire symmetric projectiles were developed based on a nine-state linear model.
- The Linear Piecewise Time Varying Technique was adapted for indirect (high pitch) fire.
- Preliminary results indicate good performance.
- Use of vacuum trajectory prediction and Pitch Perturbation state result in ***implicit*** guidance of the missile to a terminal vacuum trajectory.

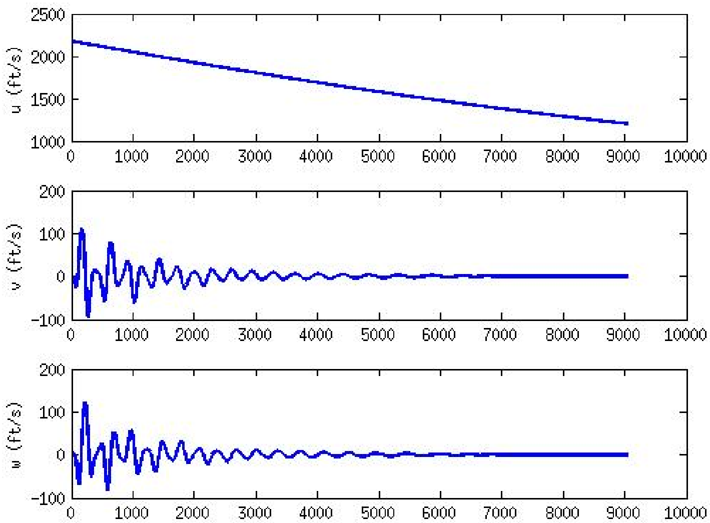
Questions

Backup slides

• Results, Linear Plant, LTI control







The World's Forum for Aerospace Leadership