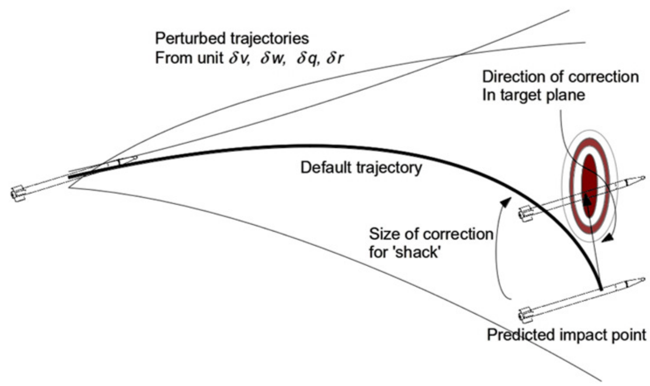

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Three Recent Advances in Optimal Control of Fin Stabilized Surface to Surface Missiles

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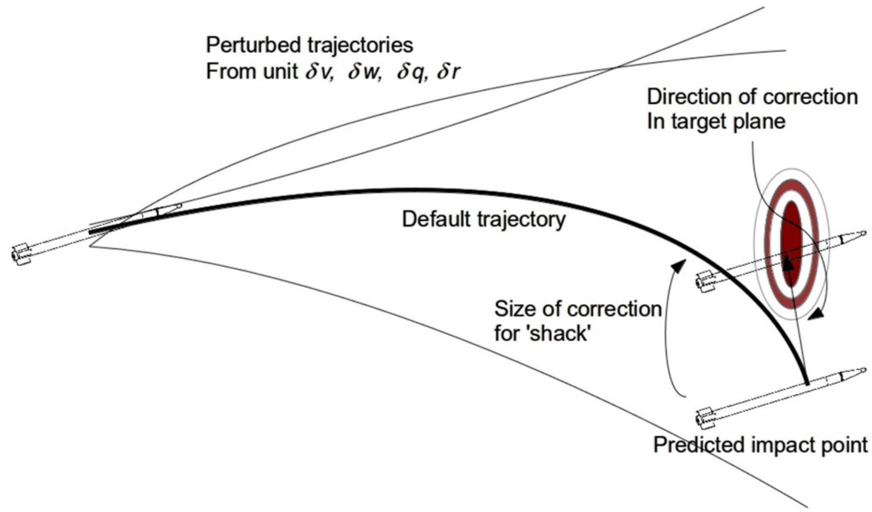
Technique #1 - Predictive Optimal Pulse-jet Control for Symmetric Projectiles



Motivation

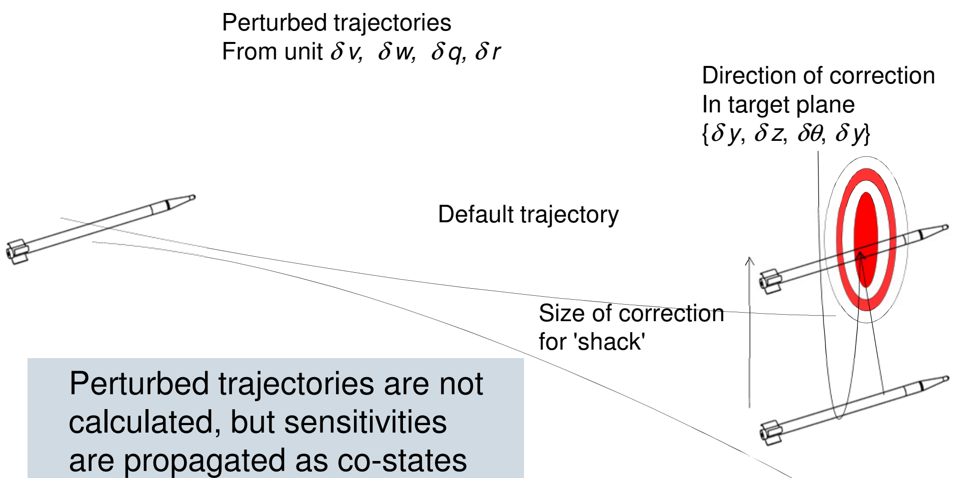
Previous efforts (2001, 2008) required a prediction of the uncontrolled, and controlled trajectories.

Pulsing changes angular rates leaving other states alone. (impulse model)

Interim work (2011) found trajectory sensitivities to changes in initial angular rates by finite differencing.

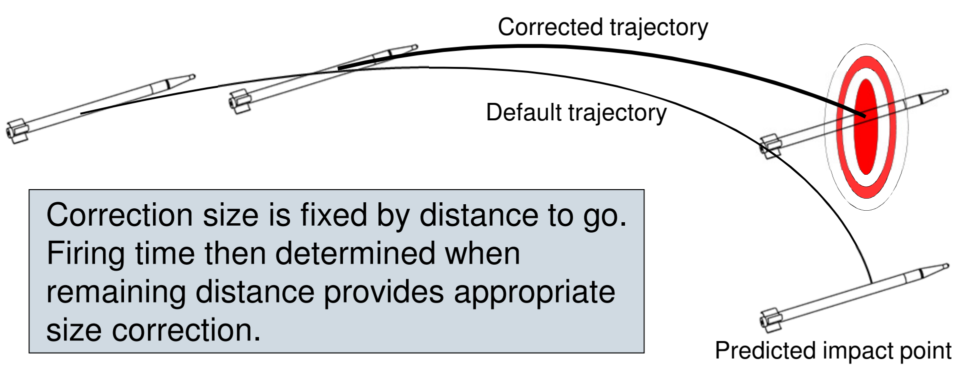
Current method uses closed-form sensitivities (2012) which are computed six times faster than finite differences.

Knowing the predicted impact point and sensitivity with respect to changes in initial angular rates, correction times are determined by two methods:



Perturbed trajectories are not calculated, but sensitivities are propagated as co-states

Target plane direction is mapped to angular rate correction vector through a Jacobian matrix. This in turn maps to roll angle at which pulse jet should be fired



Correction size is fixed by distance to go. Firing time then determined when remaining distance provides appropriate size correction.

Linear Theory Model

$$\begin{Bmatrix} y' \\ z' \\ \theta' \\ \psi' \end{Bmatrix} = \Phi \begin{Bmatrix} y \\ z \\ \theta \\ \psi \end{Bmatrix} + \frac{D}{V}\mathbf{I} \begin{Bmatrix} v \\ w \\ q \\ r \end{Bmatrix} \qquad \Phi = \begin{bmatrix} 0 & 0 & 0 & D \\ 0 & 0 & -D & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{Bmatrix} v' \\ w' \\ q' \\ r' \end{Bmatrix} = \Xi \begin{Bmatrix} v \\ w \\ q \\ r \end{Bmatrix} + \begin{Bmatrix} 0 \\ G \\ 0 \\ 0 \end{Bmatrix} g \qquad \Xi = \begin{bmatrix} -\Xi_1 & 0 & 0 & -D \\ 0 & -\Xi_1 & D & 0 \\ \Xi_2 & \Xi_3 & \Xi_4 & -\Xi_5 \\ -\Xi_3 & \Xi_2 & \Xi_5 & \Xi_4 \end{bmatrix}$$

Ξ contains epicyclic modes

Ξ *is invertible*

Ξ_2 is the Magnus (=0 in this study)

Linear Theory Solution

Velocity state total solution is written in matrix exponential form for ease of differentiation.

$$\eta = e^{\mathbf{\Xi}s}\chi + \eta_p$$

Position state solution not so easily found since Φ matrix singular.

$$\mathbf{x}'(s) = \mathbf{\Phi}\mathbf{x}(s) + \mathbf{\Gamma}(s)$$

Instead, treat as linear forced ODE and use the solution:

$$\mathbf{x}(s) = e^{\mathbf{\Phi}s}\mathbf{x}_0 + e^{\mathbf{\Phi}s}\int_0^s e^{-\mathbf{\Phi}\tau}\mathbf{\Gamma}(\tau)d\tau$$

Linear Theory Solution

The integral can be handled by a combined matrix exponential, if:

$$\boldsymbol{\Psi} = \begin{bmatrix} \boldsymbol{\Phi} & \boldsymbol{\Gamma} \\ \mathbf{0} & R \end{bmatrix} \quad \text{and} \quad e^{\boldsymbol{\Psi}s} = \begin{bmatrix} \Omega_1 & \Delta_1 \\ \mathbf{0} & \Omega_2 \end{bmatrix}$$

Then:

$$\begin{aligned} \Omega_1 &= e^{\boldsymbol{\Phi}s} \\ \Delta_1 &= \Omega_1 \int_0^s e^{-\boldsymbol{\Phi}\tau} \boldsymbol{\Gamma} e^{R\tau} d\tau \end{aligned}$$

Derivatives of the closed form solution are now found more easily.

$$\frac{\partial \xi}{\partial \vartheta_i} = \Omega_1 \frac{\partial \xi_0}{\partial \vartheta_i} + \frac{\partial \Delta_1}{\partial \vartheta_i}$$

$$\frac{\partial \Delta_1}{\partial \vartheta_i} = \int_0^s e^{-\boldsymbol{\Phi}\tau} \frac{\partial \boldsymbol{\Gamma}}{\partial \vartheta_i} d\tau$$

$$\frac{\partial}{\partial \vartheta_i}\mathbf{\Psi} = \begin{bmatrix} \mathbf{\Phi} & \frac{\partial}{\partial \vartheta_i}\mathbf{\Gamma} \\ \mathbf{0} & 0 \end{bmatrix}$$

$$\exp\left(s\frac{\partial}{\partial \vartheta_i}\mathbf{\Psi}\right) = \begin{bmatrix} \Omega_3 & \frac{\partial \Delta_1}{\partial \vartheta_i} \\ 0 & \Omega_4 \end{bmatrix}$$

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Derivatives of initial angular rates at current downrange distance:

$$\begin{aligned} \left.\frac{\partial \eta}{\partial v_0}\right|_{s=0} &= e^{\mathbf{\Xi}}\left\{\begin{array}{cccc} 1 & 0 & 0 & 0 \end{array}\right\}^T \\ \left.\frac{\partial \eta}{\partial w_0}\right|_{s=0} &= e^{\mathbf{\Xi}}\left\{\begin{array}{cccc} 0 & 1 & 0 & 0 \end{array}\right\}^T \\ \left.\frac{\partial \eta}{\partial q_0}\right|_{s=0} &= e^{\mathbf{\Xi}}\left\{\begin{array}{cccc} 0 & 0 & 1 & 0 \end{array}\right\}^T \\ \left.\frac{\partial \eta}{\partial r_0}\right|_{s=0} &= e^{\mathbf{\Xi}}\left\{\begin{array}{cccc} 0 & 0 & 0 & 1 \end{array}\right\}^T \end{aligned}$$

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This influence must be propagated along the trajectory:

$$
\begin{aligned}
\left.\frac{\partial \eta}{\partial v_0}\right|_{s=k} &= e^{\Xi} \left.\frac{\partial \eta}{\partial v_0}\right|_{s=k-1} \\
\left.\frac{\partial \eta}{\partial w_0}\right|_{s=k} &= e^{\Xi} \left.\frac{\partial \eta}{\partial w_0}\right|_{s=k-1} \\
\left.\frac{\partial \eta}{\partial q_0}\right|_{s=k} &= e^{\Xi} \left.\frac{\partial \eta}{\partial q_0}\right|_{s=k-1} \\
\left.\frac{\partial \eta}{\partial r_0}\right|_{s=k} &= e^{\Xi} \left.\frac{\partial \eta}{\partial r_0}\right|_{s=k-1}
\end{aligned}
$$

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Taylor Series Model Predictive Controller

$$
\left\{ \begin{array}{c} y_t \\ z_t \\ \theta_t \\ \psi_t \end{array} \right\} = \left\{ \begin{array}{c} \hat{y} \\ \hat{z} \\ \hat{\theta} \\ \hat{\psi} \end{array} \right\} + \left[\begin{array}{cccc} \frac{\partial y}{\partial v_0} & \frac{\partial y}{\partial w_0} & \frac{\partial y}{\partial q_0} & \frac{\partial y}{\partial r_0} \\ \frac{\partial z}{\partial v_0} & \frac{\partial z}{\partial w_0} & \frac{\partial z}{\partial q_0} & \frac{\partial z}{\partial r_0} \\ \frac{\partial \theta}{\partial v_0} & \frac{\partial \theta}{\partial w_0} & \frac{\partial \theta}{\partial q_0} & \frac{\partial \theta}{\partial r_0} \\ \frac{\partial \psi}{\partial v_0} & \frac{\partial \psi}{\partial w_0} & \frac{\partial \psi}{\partial q_0} & \frac{\partial \psi}{\partial r_0} \end{array} \right] \left\{ \begin{array}{c} \delta v_0 \\ \delta w_0 \\ \delta q_0 \\ \delta r_0 \end{array} \right\}
$$

Since pitch and yaw are unconstrained at the target, replace the last two rows with:

$$
\left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\} = \left[\begin{array}{cccc} m & 0 & 0 & -\frac{I_{yy}}{R_{Ix}} \\ 0 & m & \frac{I_{yy}}{R_{Ix}} & 0 \end{array} \right] \left\{ \begin{array}{c} \delta v_0 \\ \delta w_0 \\ \delta q_0 \\ \delta r_0 \end{array} \right\}
$$

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Euler-Lagrange Optimal Control

$$\begin{Bmatrix} y(1) \\ z(1) \end{Bmatrix} = \begin{Bmatrix} y(0) \\ z(0) \end{Bmatrix} + \begin{bmatrix} \frac{\partial y}{\partial v_0} & \frac{\partial y}{\partial w_0} & \frac{\partial y}{\partial q_0} & \frac{\partial y}{\partial r_0} \\ \frac{\partial z}{\partial v_0} & \frac{\partial z}{\partial w_0} & \frac{\partial z}{\partial q_0} & \frac{\partial z}{\partial r_0} \end{bmatrix} \begin{Bmatrix} \delta v_0 \\ \delta w_0 \\ \delta q_0 \\ \delta r_0 \end{Bmatrix}$$

$$\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^T\lambda$$

$$\begin{Bmatrix} \delta v_0 \\ \delta w_0 \\ \delta q_0 \\ \delta r_0 \end{Bmatrix} = -\frac{1}{\gamma} \begin{bmatrix} \frac{\partial y}{\partial v_0} & \frac{\partial z}{\partial v_0} \\ \frac{\partial y}{\partial w_0} & \frac{\partial z}{\partial w_0} \\ \frac{\partial y}{\partial q_0} & \frac{\partial z}{\partial q_0} \\ \frac{\partial y}{\partial r_0} & \frac{\partial z}{\partial r_0} \end{bmatrix} \begin{Bmatrix} y(1) \\ z(1) \end{Bmatrix}$$

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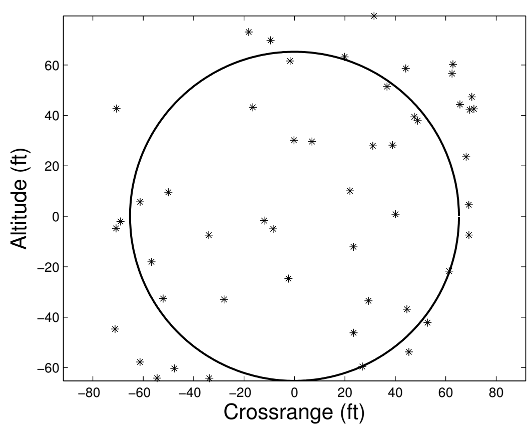
Uncontrolled Dispersion



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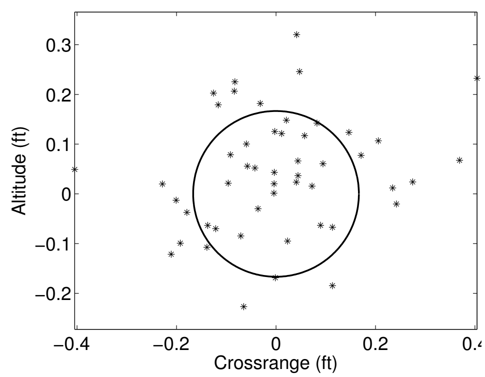
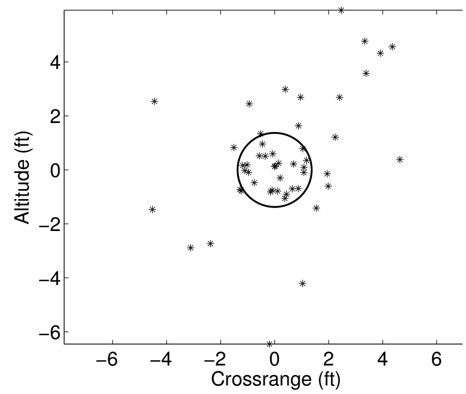
Controlled Dispersions

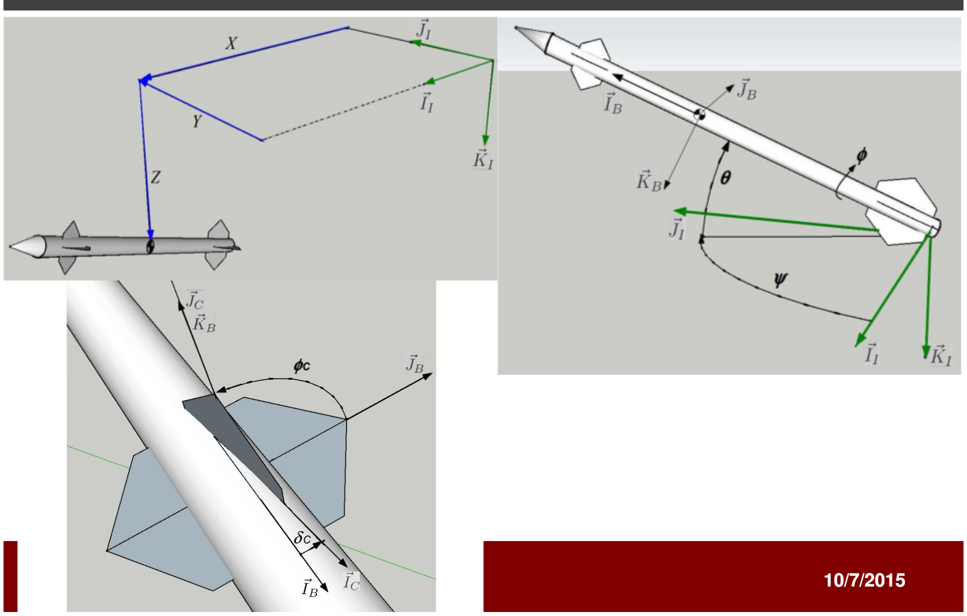


Taylor Series



One Step Optimal

Technique #2 - Euler-Lagrange Optimal Control for Direct Fire



Nine State Linear Plant Model

$$\left\{\begin{array}{c}\dot{\xi}\\ \dot{\eta}\\ \ddot{w}\end{array}\right\} = \begin{bmatrix}\mathbf{\Phi} & \mathbf{\Gamma} & \mathbf{0}\\ \mathbf{0} & \mathbf{\Xi} & \mathbf{\Lambda}\\ \mathbf{0} & \mathbf{0} & 0\end{bmatrix}\left\{\begin{array}{c}\xi\\ \eta\\ \dot{w}\end{array}\right\} + \begin{bmatrix}\mathbf{0}\\ \mathbf{b}\\ \mathbf{0}\end{bmatrix}\left\{\begin{array}{c}C_{Z0}\\ C_{Y0}\end{array}\right\}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\xi = \begin{bmatrix} y & z & \theta & \psi \end{bmatrix}^T,\ \eta = \begin{bmatrix} v & w & q & r \end{bmatrix}^T,\ \mathbf{\Lambda} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T,\ \mathbf{\Gamma} = \frac{D}{V}\mathbf{I}$$

$$\mathbf{\Phi} = \begin{bmatrix} 0 & 0 & 0 & D\\ 0 & 0 & -D & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix},\ \mathbf{\Xi} = \begin{bmatrix} -\Xi_1 & 0 & 0 & -D\\ 0 & -\Xi_1 & D & 0\\ \Xi_2 & \Xi_3 & \Xi_4 & -\Xi_5\\ -\Xi_3 & \Xi_2 & \Xi_5 & \Xi_4 \end{bmatrix}$$

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LTI Finite Horizon Optimal Control

System matrices treated as constants such that **A**(p, V) = **A**, etc.:

Form the finite horizon cost function:

$$J = \frac{1}{2}\mathbf{x}^T(s_t)\mathbf{P}\mathbf{x}(s_t) + \int_{s_i}^{s_t} \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\,ds$$

Since Q ≥ 0, choose Q = 0, then define the Hamiltonian:

$$\tilde{H} = \frac{1}{2}\mathbf{u}^T\mathbf{R}\mathbf{u} + \lambda^T(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u})$$

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LTI Finite Horizon Optimal Control, Cont'd

Taking variations yields the conditions for optimality:

$$\begin{aligned} \dot{\lambda}^T &= -\frac{\partial \tilde{H}}{\partial \mathbf{x}} = -\lambda^T \mathbf{A} \\ \frac{\partial \tilde{H}}{\partial \mathbf{u}} &= \mathbf{0} \rightarrow \mathbf{0} = \mathbf{u}^T \mathbf{R} + \lambda^T \mathbf{B} \end{aligned}$$

Thus the control law is

$$\mathbf{u} = -\mathbf{R}^{-1} \mathbf{B}^T \lambda$$

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LTI Finite Horizon Optimal Control, Cont'd

State and co-state ODEs are collected into a single matrix equation:

$$\left\{ \begin{array}{c} \dot{\mathbf{x}} \\ \dot{\lambda} \end{array} \right\} = \left[\begin{array}{cc} \mathbf{A} & -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T \\ \mathbf{0} & -\mathbf{A}^T \end{array} \right] \left\{ \begin{array}{c} \mathbf{x} \\ \lambda \end{array} \right\}$$

Which can be solved using state transition matrices such that:

$$\left\{ \begin{array}{c} \mathbf{x}(s) \\ \lambda(s) \end{array} \right\} = \left[\begin{array}{cc} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{array} \right] \left\{ \begin{array}{c} \mathbf{x}(s_t) \\ \mathbf{P}\mathbf{x}(s_t) \end{array} \right\}$$

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LTI Optimal Control cont'd

Such that the current co-state is given by:

$$\lambda(s) = [\mathbf{\Sigma}_{21} + \mathbf{\Sigma}_{22}\mathbf{P}] \bullet [\mathbf{\Sigma}_{11} + \mathbf{\Sigma}_{12}\mathbf{P}]^{-1}\mathbf{x}(s)$$

And the transition matrices are given by the full 18 x 18 matrix exponential:

$$\begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{bmatrix} = \exp\begin{bmatrix} \mathbf{A}\sigma & -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\sigma \\ \mathbf{0} & -\mathbf{A}^T\sigma \end{bmatrix}$$

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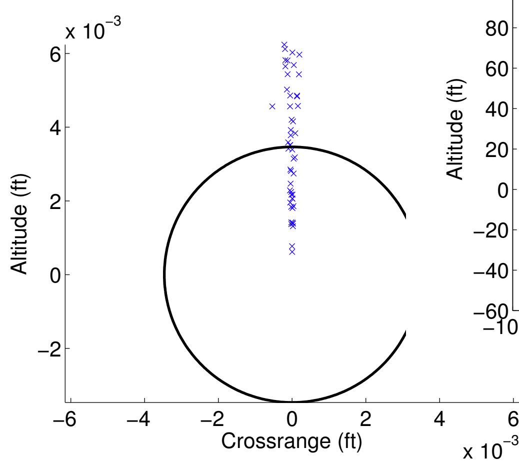
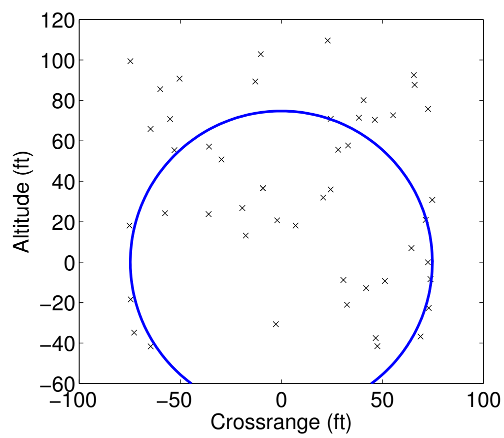
Dispersions, uncontrolled, controlled



Controlled, no range correction



Uncontrolled

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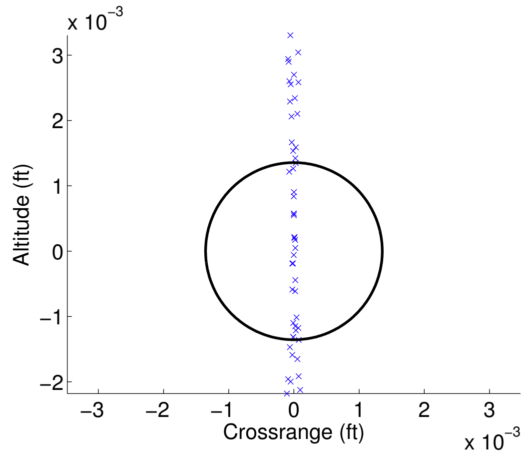
Dispersion, with range correction



Controlled, with range correction

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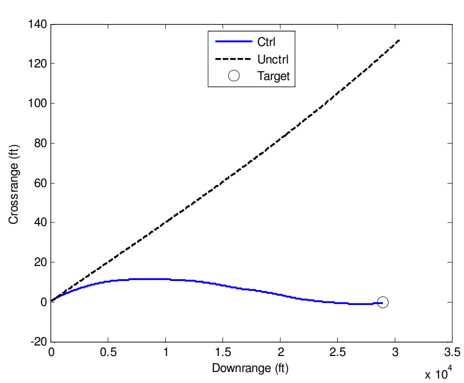
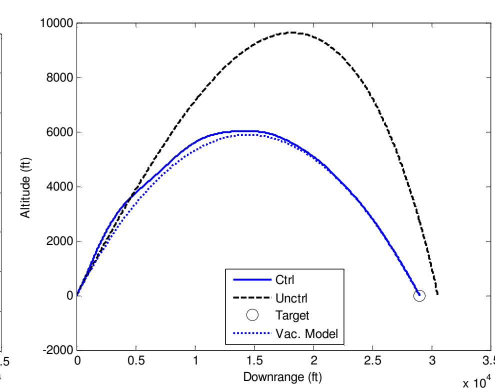
Technique #3 - Euler-Lagrange Optimal Control for Indirect Fire





Modified Projectile Linear Theory Equations in Matrix Form

$$\begin{Bmatrix} \dot{\boldsymbol{\xi}} \\ \dot{\boldsymbol{\eta}} \\ \ddot{w} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{\Phi} & \boldsymbol{\Gamma} & \boldsymbol{\Sigma} \\ 0 & \boldsymbol{\Xi} & \boldsymbol{\Lambda} \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \\ \dot{w} \end{Bmatrix} + \begin{bmatrix} 0 \\ \mathbf{b} \\ 0 \end{bmatrix} \begin{Bmatrix} C_{Z0} \\ C_{Y0} \end{Bmatrix}$$

$$\boldsymbol{\Sigma} = D s_{\bar{\theta}} [0 \;\; -1 \;\; 0 \;\; 0]^T \;, \boldsymbol{\Lambda} = \frac{Dg}{V} c_{\bar{\theta}} [0 \;\; 1 \;\; 0 \;\; 0]^T$$

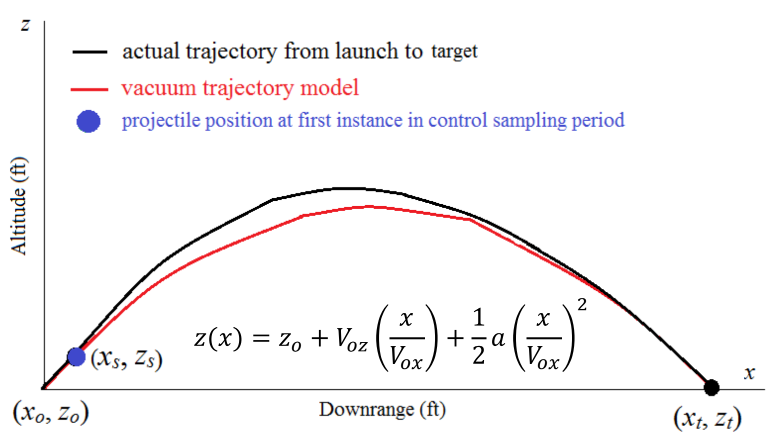
$$\boldsymbol{\eta} = [v \;\; w \;\; q \;\; r]^T \qquad \boldsymbol{\xi} = [y \;\; z \;\; \delta_\theta \;\; \psi]^T$$

Pitch angle, roll rate, and total velocity are treated as parameters rather than states

Pitch angle perturbation is added to the state vector to preserve controllability

Angle of attack rate $\dot{w}$ is uncontrollable but stabiliziable state s.t. gravity is included in homogeneous solution

Point mass vacuum trajectory is contrived to intersect early point in actual trajectory, and target.



Time varying parameters (V, p, sinθ, cosθ) are predicted from point mass vacuum trajectory.

$$V(s+h)=\sqrt{\left(V^2(s)+\frac{b_v}{a_v}\right)e^{-2a_v h}-\frac{b_v}{a_v}}$$

$$p(s+h)=C_{pe1}e^{C_{pe2}h}-C_{p0}$$

$$c_{\theta_k}=\frac{dx}{ds_k} \qquad s_{\theta_k}=\frac{dz_k}{ds_k}$$

$$V(s+h)=\sqrt{\left(V^2(s)+\frac{b_v}{a_v}\right)e^{-2a_v h}-\frac{b_v}{a_v}}$$

Euler-Lagrange Optimal solution for Time Varying Piecewise Linear systems

$$J=\frac{1}{2}\mathbf{x}^T(s_t)\mathbf{P}\mathbf{x}(s_t)+\int_{s_i}^{s_t}\frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x}+\mathbf{u}^T\mathbf{R}\mathbf{u}\,ds$$

The control can be found from:

$$\mathbf{u}(s)=-\mathbf{R}^{-1}\mathbf{B}^T(s)\mathbf{N}(s)\mathbf{x}(s)$$

N(*s*) is the solution to the time varying Riccati eqn:

$$\dot{\mathbf{N}}(s)=-\mathbf{N}(s)\mathbf{A}(s)-\mathbf{A}^T(s)\mathbf{N}(s)+\mathbf{N}(s)\mathbf{B}(s)\mathbf{R}^{-1}\mathbf{B}^T(s)\mathbf{N}(s)-\mathbf{Q}$$

Decompose the Riccati Eqn. into two DEs:

$$\begin{Bmatrix} \dot{\mathbf{W}}(s) \\ \dot{\mathbf{Y}}(s) \end{Bmatrix} = \begin{bmatrix} \mathbf{A}(s) & -\mathbf{B}(s)\mathbf{R}^{-1}\mathbf{B}^T(s) \\ 0 & -\mathbf{A}^T(s) \end{bmatrix} \begin{Bmatrix} \mathbf{W}(s) \\ \mathbf{Y}(s) \end{Bmatrix}$$

$$\dot{\mathbf{Z}}(s) = \mathbf{F}(s)\mathbf{Z}(s)$$

$$\mathbf{N}(s) = \mathbf{Y}(s)\mathbf{W}(s)^{-1}$$

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Set **Z**(*s*) at target plane:

$$\mathbf{W}(s_t) = \mathbf{I}$$

$$\mathbf{Y}(s_t) = \mathbf{P}$$

Back propagate the solution from target plane to current downrange arclength:

$$\mathbf{Z}_{n_s} = \left(\mathbf{I} + \frac{h}{2}\mathbf{F}_{n_s}\right)^{-1} \mathbf{Z}(s_t)$$

$$\mathbf{Z}_k = \left(\mathbf{I} + \frac{h}{2}\mathbf{F}_k\right)^{-1} \left(\mathbf{I} - \frac{h}{2}\mathbf{F}_{k+1}\right) \mathbf{Z}_{k+1},\ k = 0, 1, \dots, n_s - 1$$

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Algorithm reviewed:

- At the first time in the control sampling period, solve for and save the coefficients for creating the vacuum model.
  - The model will launch from the origin, intersect a projectile state early in the trajectory, and hit the target.
- Compute the pitch angle to get from current position to the first spot on the vacuum trajectory. From the prediction of this angle, develop the current value of the δ_θ state.
- Recursively predict values for p, V, c_θ, s_θ, and h using the vacuum trajectory model while updating aerodynamic coefficients at each segment based on new predicted velocity and altitude.
- Build the corresponding matrix Hamiltonian for each segment.
- Integrate backwards in time using (54) – (55).
- Using Z_1, compute the Riccati solution at the current state from (52).
- Compute the control needed at the current state using (47).
- Convert to dimensional form, limit from [-1,1] rad and rotate into roll frame.

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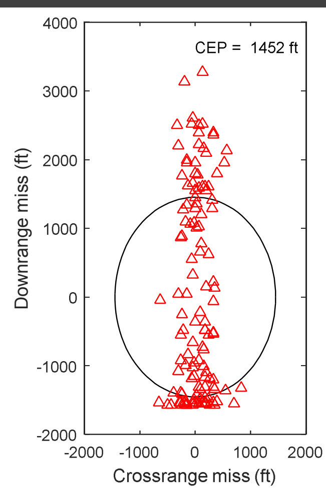
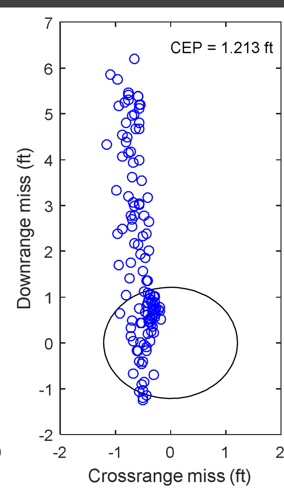
Results



Uncontrolled



Controlled

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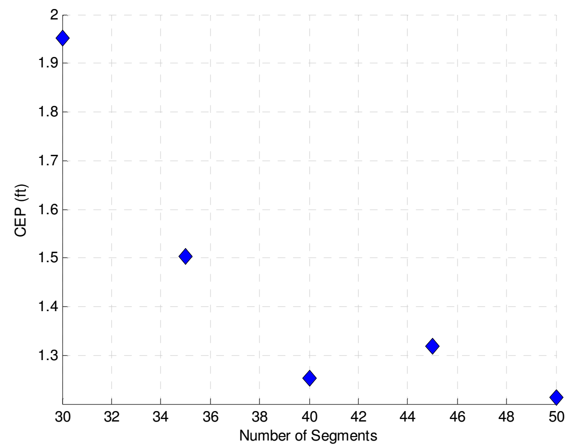
- Nash, A. L.,†‡ and Burchett, B. T., "Euler-Lagrange Optimal Control of Indirect Fire Symmetric Projectiles", Accepted for presentation at *AIAA Science and Technology Forum and Exposition 2016,* San Diego, CA, 4-8 January, 2016.
- Burchett, B. T., and Nash, A. L.,† "Euler-Lagrange Optimal Control for Symmetric Projectiles", *Proceedings of the AIAA Science and Technology Forum and Exposition 2015,* Kissimee, FL, 5-9 January, 2015. doi: 10.2514/6.2015-1020
- Burchett, B. T., "Predictive Optimal Pulse-jet Control for Symmetric Projectiles", *Proceedings ofthe AIAA Science and Technology Forum and Exposition 2014,* National Harbor, MD, AIAA paper no. 2014-0883. doi: 10.2514/6.2014-0883

Backup Slides

Trade Study – Effect of Trajectory Discretization



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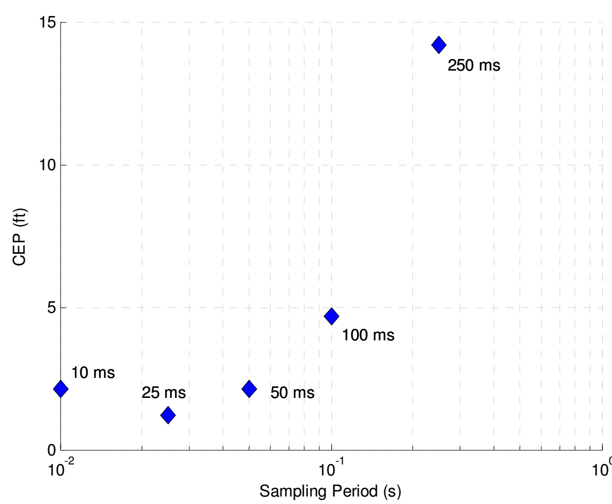
Trade Study – Effect of Controller Sampling Period



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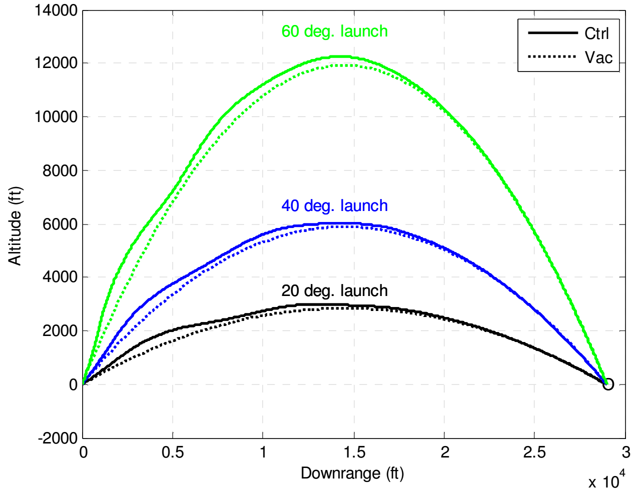
Robustness of Guidance – Convergence to Vacuum Trajectory



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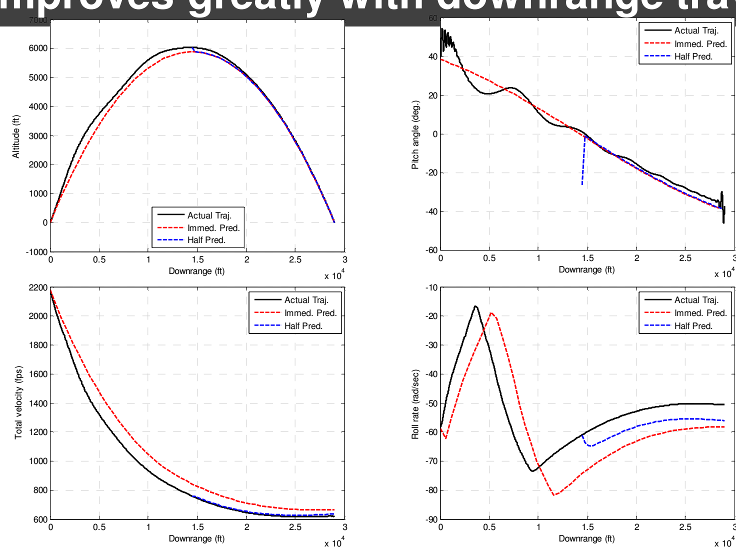
Limited predictions required – Accuracy improves greatly with downrange travel.



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