Rose-**H**ulman **I**nstitute of **T**echnology
Department of Mechanical Engineering

ME 536 | Comput Intell in Control Engr

Homework 2: Deadline: March 27, 2020
Course Value: 100 points

a) Advanced Numerical methods Part I: Determining unknown initial conditions from the time history of a dynamic system.

For certain dynamic systems, the system initial state may not be readily known from the time history of system outputs. In these cases, the initial state vector may be estimated by a gradient least-squares search, where the sensitivity of system states at a specific time is related to the system initial state by propagating a set of costates forward in time. In general, the dynamics of the costates is given by

$$\frac{d}{dt}\left(\frac{\partial \mathbf{X}}{\partial \mathbf{X}_0}\right) = \frac{\partial \dot{\mathbf{X}}}{\partial \mathbf{X}}\frac{\partial \mathbf{X}}{\partial \mathbf{X}_0} \quad .$$

For this assignment, you will estimate the non-zero initial velocity for the swinging pendulum simulated in Homework 1, such that the pendulum swings to vertical in the time measured experimentally. Knowing the angular velocity as the pendulum crosses the horizontal position, you can then use your backward RK4 code to estimate an initial position slightly above horizontal from which, if the pendulum were released from rest, it would gain the required angular velocity as it crossed the horizontal position.

First, you will modify your RK4 code from Homework 1a to propagate four costates forward in time. Let's derive the equations for the costates. First, express the dynamics of the pendulum in state space form

$$J\ddot{\theta} + k\sin\theta = 0 \;, \tag{1}$$

Becomes

$$\begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix} = \begin{Bmatrix} \theta_2 \\ -\dfrac{k}{J}\sin\theta_1 \end{Bmatrix} \tag{2}$$

Then write the costates DE in matrix form in terms of θ_1 and θ_2.

$$\frac{d}{dt}\left(\frac{\partial \mathbf{\Theta}}{\partial \mathbf{\Theta}_0}\right) = \frac{\partial \dot{\mathbf{\Theta}}}{\partial \mathbf{\Theta}}\frac{\partial \mathbf{\Theta}}{\partial \mathbf{\Theta}_0} \quad . \tag{4}$$

Which expands to:

$$\frac{d}{dt}\left(\frac{\partial \mathbf{\Theta}}{\partial \mathbf{\Theta}_0}\right) = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{J}\cos\theta_1 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{\partial \theta_1}{\partial \theta_{10}} & \dfrac{\partial \theta_1}{\partial \theta_{20}} \\ \dfrac{\partial \theta_2}{\partial \theta_{10}} & \dfrac{\partial \theta_2}{\partial \theta_{20}} \end{bmatrix} \quad . \tag{5}$$

Completing the multiplication, we have

$$\frac{d}{dt}\left(\begin{bmatrix} \frac{\partial\theta_1}{\partial\theta_{10}} & \frac{\partial\theta_1}{\partial\theta_{20}} \\ \frac{\partial\theta_2}{\partial\theta_{10}} & \frac{\partial\theta_2}{\partial\theta_{20}} \end{bmatrix}\right) = \begin{bmatrix} \frac{\partial\theta_2}{\partial\theta_{10}} & \frac{\partial\theta_2}{\partial\theta_{20}} \\ -\frac{k}{J}\cos\theta_1\frac{\partial\theta_1}{\partial\theta_{10}} & -\frac{k}{J}\cos\theta_1\frac{\partial\theta_1}{\partial\theta_{20}} \end{bmatrix}. \tag{6}$$

You will need to code the append the four scalar equations from (6) to your RK4 code which uses (2). The initial conditions for the costates of (6) are simply

$$\left.\begin{bmatrix} \frac{\partial\theta_1}{\partial\theta_{10}} & \frac{\partial\theta_1}{\partial\theta_{20}} \\ \frac{\partial\theta_2}{\partial\theta_{10}} & \frac{\partial\theta_2}{\partial\theta_{20}} \end{bmatrix}\right|_{\mathrm{t=0}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{7}$$

Integrate all 6 states forward in time for $0 \leq t \leq t_f$, assuming the pendulum is released from rest at $\theta_0 = -\pi/2$. Use $L_{\mathrm{w,cg}} = 3.75$ and $t_f = 0.251$ sec as placeholders until completing part b.

b) Advanced Numerical part II: Use your code from part a as the function and gradient evaluation in a one-dimensional Newton algorithm to find the initial angular rate of the pendulum, such that the time to swing from horizontal to vertical matches the following experimental data set:

$$L_{\mathrm{w,cg}} = 3.75 + 5n \text{ cm}, n = 0 \ldots 6.$$

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tf = [0.251, 0.239, 0.240, 0.251, 0.264, 0.284, 0.297];
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Note that the residual will be the pendulum angle at the final time (since $\theta_f = 0$ is desired). The gradient will be

$$\frac{\partial\theta_1}{\partial\theta_{20}}$$

Such that the Newton iteration will be

$$\theta_{20}^{new} = \theta_{20}^{old} - \frac{\Delta\theta_1}{\frac{\partial\theta_1}{\partial\theta_{20}}}$$

c) Advanced Numerical methods part III: Use your backward integration code from Homework 1b to estimate the initial angle θ_0 such that, if released from rest, the pendulum would reach the angular velocity determined in part b when it crosses the horizontal.

Submit a table with your results from parts b and c, and plots to support the numbers.