The Wave Equation IV

The Nonhomogeneous Case
MA 436
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Introduction

Consider the non-homogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = F(x,t) \tag{1}$$

for a function $u(x,t)$ for $-\infty < x < \infty$ and $t > 0$. Here the function F can be thought of physically as a "force per unit length", divided by linear density, acting on the string (go back to the derivation of the wave equation to see this; F has units of acceleration, and can be though of as the acceleration induced in the string by an applied force). We want to solve the non-homogeneous equation with given initial conditions.

In what follows we'll seek a solution to equation (1) with ZERO initial conditions. This makes it easy to solve the equation with any desired initial conditions—just use linearity and superposition: If we find a solution $u_1(x,t)$ to equation (1) with $u_1(x,0) = \frac{\partial u_1}{\partial t}(x,0) = 0$, and a solution $u_2(x,t)$ to the homogeneous wave equation with $u_2(x,0) = f(x)$ and $\frac{\partial u_2}{\partial t}(x,0) = g(x)$ (which we now how to do), then $u(x,t) = u_1(x,t) + u_2(x,t)$ satisfies (1) with $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$.

In what follows we'll use the following facts from Calculus:

$$\begin{aligned}
\frac{d}{dx}\int_a^x f(z)\,dz &= f(x) &(2)\\
\frac{d}{dx}\int_x^a f(z)\,dz &= -f(x) &(3)\\
\frac{d}{dx}\int_a^x f(x,z)\,dz &= f(x,x) + \int_a^x f_x(x,z)\,dz &(4)\\
\frac{d}{dx}\int_x^a f(x,z)\,dz &= -f(x,x) + \int_a^x f_x(x,z)\,dz. &(5)
\end{aligned}$$

Change of Variables

Here's a clever trick for finding a solution to equation (1). We do a change coordinates. Specifically, let $y = x - ct$ and $s = x + ct$ (so $x = (s+y)/2, t = (s-y)/(2c)$). This new coordinate system can be visualized as two new axes: a s axis (corresponding to $y = 0$) which runs along the line $x = ct$, and a y axis (corresponding to $s = 0$) running along $x = -ct$. Also note that we'll always have $y \leq s$ (with $y = s$ corresponding to $t = 0$):

Define a function $v(y,s) = u(x,t)$ (so v is just u, but in the new coordinate system. We can write, more explicitly

$$u(x,t) = v(y,s) = v(x-ct, x+ct).$$

We find that (use the chain rule)

$$\begin{aligned} u_t(x,t) &= -cv_y(x-ct,x+ct) + cv_s(x-ct,x+ct), \\ u_x(x,t) &= v_y(x-ct,x+ct) + v_s(x-ct,x+ct). \end{aligned}$$

The second derivatives are

$$\begin{aligned} u_{tt}(x,t) &= c^2v_{yy}(x-ct,x+ct) - 2c^2v_{sy}(x-ct,x+ct) + c^2v_{ss}(x-ct,x+ct), \quad (6) \\ u_{xx}(x,t) &= v_{yy}(x-ct,x+ct) + 2v_{sy}(x-ct,x+ct) + v_{ss}(x-ct,x+ct). \quad (7) \end{aligned}$$

If $u_{tt} - c^2u_{xx} = F$ then we find from the above equations (6)-(7) that, amazingly

$$-4c^2v_{sy}(x-ct,x+ct) = F(x,t)$$

or, using $y = x - ct$, $s = x + ct$, just (back to Leibnitz notation)

$$\frac{\partial^2 v}{\partial s \partial y}(y,s) = -\frac{1}{4c^2}G(y,s). \tag{8}$$

where $G(y,s) = F(x,t) = F(\frac{s+y}{2}, \frac{s-y}{2c})$ denotes F in the (y,s) coordinate system. Equation (8) is the wave equation, but in the new (y,s) coordinate system.

Example: The Homogeneous Case

Let's take a moment to consider the case $F \equiv 0$ (so $G \equiv 0$). In this case equation (8) becomes just

$$\frac{\partial^2 v}{\partial s \partial y}(y,s) = 0. \tag{9}$$

This is easy to solve: Integrate both sides in s to find that

$$\frac{\partial v}{\partial y}(y,s) = c(y) \tag{10}$$

where $c(y)$ indicates a "constant" with respect to s (which can be a function of y). Now integrate both sides of equation (10) with respect to y to find

$$v(y,s) = c_1(y) + c_2(s) \tag{11}$$

where $c_1(y) = \int c(y)\,dy$ and $c_2(s)$ is some "constant" with respect to y. (You should plug v from equation (11) back into equation (9) to make sure you belief this is in fact a solution!) If we switch back to (x,t) coordinates and recall that $u(x,t) = v(y,s)$ with $y = x - ct, s = x + t$ then we have

$$u(x,t) = c_1(x - ct) + c_2(x + ct)$$

for some functions c_1 and c_2. This shows (as we did prior to the D'Alembert formula) that the solution consists of a right moving ($c_1(x - ct)$) and left moving ($c_2(x + ct)$) wave. We can then rig c_1 and c_2 as before to get desired initial conditions.

Back to the Nonhomogeneous Case

We can proceed pretty much as in the homogeneous case, but I'll use definite integrals and be careful with limits. Integrate both sides of equation (8) with respect to y; the upper limit should be $y = y_0$, but lower limit can be ANY function of s, say $A(s)$. You obtain

$$\frac{\partial v}{\partial s}(y_0, s) - \frac{\partial v}{\partial s}(A(s), s) = -\frac{1}{4c^2}\int_{A(s)}^{y_0} G(y,s)\,dy.$$

Let's move the $\frac{\partial v}{\partial s}(A(s), s)$ to the right side and just call it $c(s)$, to obtain

$$\frac{\partial v}{\partial s}(y_0, s) = -\frac{1}{4c^2}\int_{A(s)}^{y_0} G(y,s)\,dy + c(s). \tag{12}$$

You can think of $c(s)$ as a "constant of integration", at least constant in y. If you're suspicious of this, differentiate both sides of equation (12) in y_0 and confirm that you get back (8), no matter what you take for $c(s)$.

Now integrate both sides of equation (12) with respect to s from $s = B(y_0)$ to $s = s_0$ (where $B(y_0)$ is some function of y_0) to find

$$v(y_0, s_0) - v(y_0, B(y_0)) = -\frac{1}{4c^2}\int_{B(y_0)}^{s_0}\int_{A(s)}^{y_0} G(y,s)\,dy\,ds + C(s_0) - C(B(y_0))$$

where $C(s)$ is an anti-derivative for $c(s)$. Move the $v(y_0, B(y_0))$ term to the right side and write the whole thing as

$$v(y_0, s_0) = -\frac{1}{4c^2}\int_{B(y_0)}^{s_0}\int_{A(s)}^{y_0} G(y,s)\,dy\,ds + c_1(s_0) + c_2(y_0)$$

for some functions $c_1(s_0)$ and $c_2(y_0)$. Again, you can differentiate both sides of the above equation in s_0 and verify you get back (12).

Now notice that if we were to back to the (x,t) coordinate system, $c_1(s)$ and $c_2(y)$ would become $c_1(x + ct)$ and $c_2(x - ct)$, respectively, which are solutions to the homogeneous wave equation. Let's drop them, since they don't contribute to the nonhomogeneous term we're trying to obtain (but they do come in to play for the initial conditions, though we won't need them). We're left with

$$v(y_0, s_0) = -\frac{1}{4c^2} \int_{B(y_0)}^{s_0} \int_{A(s)}^{y_0} G(y, s)\, dy\, ds. \tag{13}$$

You can check that v as defined by equation (13) satisfies the PDE (8) for any choice of $A(s)$ and $B(y_0)$, so let's try to rig these functions so that v yields a function $u(x, t)$ with zero initial conditions.

Getting Zero Initial Conditions

Note that $t = 0$ in the (x, t) coordinates corresponds to $y = s$ in the (y, s) coordinate system. We'd like $u(x, 0) = 0$ for all x, or equivalently, $v(s, s) = 0$ for all s. One easy way to obtain this is to take $B(y) = y$ so that from equation (13)

$$v(y_0, s_0) = -\frac{1}{4c^2} \int_{y_0}^{s_0} \int_{A(s)}^{y_0} G(y, s)\, dy\, ds. \tag{14}$$

If $y_0 = s_0$ then $v(y_0, s_0) = 0$ is automatic.

Now we can try to pick $A(s)$ to get the condition $u_t(x, 0) = 0$. We computed $u_t(x, t) = -cv_y(x - ct, x + ct) + cv_s(x - ct, x + ct)$ on page 2, and so $u_t(x, 0) = -cv_y(x, x) + cv_s(x, x)$. Requiring $u_t(x, 0) = 0$ leads to the requirement $v_y = v_s$ on the line $y = s$. It's easy to compute

$$\frac{\partial v}{\partial s_0}(y_0, s_0) = -\frac{1}{4c^2} \int_{A(s)}^{y_0} G(y, s_0)\, dy. \tag{15}$$

To compute $\frac{\partial v}{\partial y_0}$ I'll use some of the rules (2)-(5) from the front page. Specifically, let's write

$$v(y_0, s_0) = -\frac{1}{4c^2} \int_{y_0}^{s_0} H(y_0, s)\, ds$$

where $H(y_0, s) = \int_{A(s)}^{y_0} G(y, s)\, dy$. We can compute (use equation (11))

$$\frac{\partial v}{\partial y_0}(y_0, s) = -\frac{1}{4c^2}(-H(y_0, y_0) + \int_{y_0}^{s_0} H_y(y_0, s)\, ds).$$

But by rule (2) we have $H_y(y_0, s) = G(y_0, s)$. Use this in the above equation and fill back in the definition of H to obtain

$$\frac{\partial v}{\partial y_0}(y_0, s_0) = \frac{1}{4c^2}(\int_{A(y_0)}^{y_0} G(y, y_0)\, dy - \int_{y_0}^{s_0} G(y_0, s)\, ds). \tag{16}$$

Now use equations (15) and (16) to write out the requirement that $v_s(y_0, y_0) - v_y(y_0, y_0) = 0$ for all y_0, to obtain

$$\frac{1}{4c^2} \left(-\int_{A(y_0)}^{y_0} G(y, y_0)\, dy + \int_{y_0}^{y_0} G(y_0, s)\, ds - \int_{A(y_0)}^{y_0} G(y, y_0)\, dy \right) = 0.$$

or (since the middle integral is clearly zero and the other two are identical)

$$\int_{A(y_0)}^{y_0} G(y, y_0)\, dy = 0$$

if we multiply by $1/(4c^2)$. This last equation clearly is satisfied if (and generally, only if) $A(y_0) = y_0$. So we've shown that we should take

$$v(y_0, s_0) = -\frac{1}{4c^2} \int_{y_0}^{s_0} \int_{s}^{y_0} G(y, s)\, dy\, ds.$$

One last modification: In the double integral above we have from the outer integral limits that $y_0 \leq s \leq s_0$, while the inner integral runs from $y = s$ to $y = y_0$—but $y_0 \leq s$, so let's flip the limits in the inner integral and put a minus in front. We have

$$v(y_0, s_0) = \frac{1}{4c^2} \int_{y_0}^{s_0} \int_{y_0}^{s} G(y, s)\, dy\, ds. \tag{17}$$

In some sense we could quit here: Since we defined $v(y, s) = u(x, t)$ (with $y = x - ct, s = x + ct$) we have $u(x_0, t_0) = v(x_0 - ct_0, x_0 + ct_0)$, or

$$u(x_0, t_0) = \frac{1}{4c^2} \int_{x_0 - ct_0}^{x_0 + ct_0} \int_{x_0 - ct_0}^{s} F(\frac{s+y}{2}, \frac{s-y}{2c})\, dy\, ds \tag{18}$$

where we've use the fact that $G(y, s) = F(\frac{s+y}{2}, \frac{s-y}{2c})$. Equation (18) defines the solution to (1) with zero initial conditions. But there's some value in changing the double integral back into the original (x, t) coordinate system.

Back to the Original Coordinates

The last step to finish this off is to change variables in the integral. Consider a specific point with coordinates (x_0, t_0) in the (x, t) coordinate system. Such a point has coordinates (y_0, s_0) in the (y, s) coordinate system. If you've had advanced calculus you ought to recall that we need to change the change of coordinates dictates a change in the integrand, as $dx\, dt = J dy\, ds$ where J is the Jacobian of the transformation taking (y, s) to (x, t), given by the absolute value of the determinant of the matrix

$$\begin{bmatrix} \frac{\partial x}{\partial y} & \frac{\partial x}{\partial s} \\ \frac{\partial t}{\partial y} & \frac{\partial t}{\partial s} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2c} \\ \frac{1}{2} & -\frac{1}{2c} \end{bmatrix}$$

and turns out to be $J = \frac{1}{2c}$. In short, $dx\, dt = \frac{1}{2c} dy\, ds$, or $dy\, ds = 2c\, dx\, dt$. The double integral will look like

$$\frac{1}{2c} \int_{?}^{?} \int_{?}^{?} F(x, t)\, dx\, dt$$

where the hard part is finding the limits of integration. Refer to the figures below, in which I label the point (x_0, t_0) in both coordinates systems, on the left. The double

integral in equation (18) has limits $s = y_0$ to $s = s_0$ (outer integral) and $y = y_0$ to $y = s$ (inner integral), corresponding to the shaded triangular region in the figure on the right. The three sides are given by the equations as follows: The sloped sides is $y = s$, corresponding to $t = 0$ in the original coordinates; the horizontal side is $s = s_0$, or $x + ct = x_0 + ct_0$ in original coordinates; the vertical side is $y = y_0$, or $x - ct = x_0 - ct_0$ in original coordinates. But these form exactly the sides of the backward light cone for (x_0, t_0)! So the integral for u becomes, after changing the limits of integration

$$u(x_0, t_0) = \frac{1}{2c} \int_0^{t_0} \int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} F(x, t)\, dx\, dt.$$

All in all, the solution to equation (1) with initial conditions $u(x, 0) = f(x), u_t(x, 0) = g(x)$ is given by

$$u(x_0, t_0) = \frac{1}{2}(f(x_0 - ct_0) + f(x_0 + ct_0)) + \frac{1}{2c} \int_{x_0 - ct_0}^{x_0 + ct_0} g(z)\, dz + \frac{1}{2c} \int_0^{t_0} \int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} F(x, t)\, dx\, dt.$$

One last remark: the double integral involving F is yet another confirmation of the principal of causality: only events in the backward light cone affect the solution at any given point.