The Wave Equation III

MA 436
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1 Review

We've figured out how to solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

for a function $u(x,t)$ for $-\infty < x < \infty$ and $t > 0$ with initial conditions

$$u(x,0) = f(x), \tag{2}$$
$$\frac{\partial u}{\partial t}(x,0) = g(x) \tag{3}$$

and showed that the problem is well-posed. The solution is in fact

$$u(x,t) = \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz. \tag{4}$$

Solution on a Half-Space

It's not hard to modify the solution procedure to solve the wave equation (1) on the half-line $x > 0$; think of a string with one end tied at $x = 0$. In fact, we need one additional condition: the value of u (the vertical displacement) at $x = 0$. For simplicity we'll use $u(0,t) = 0$ for all t. Of course f and g are given only for $x \geq 0$. For consistency we should assume that $f(0) = 0$ and $g(0) = 0$ (think physically: why?)

To solve the half-line problem, we extend f as an odd function to the whole real line, as

$$\tilde{f}(x) = \begin{cases} f(x), & x \geq 0 \\ -f(-x), & x < 0 \end{cases}$$

Note that $\tilde{f}$ is continuous if f is continuous. Let $\tilde{g}$ denote the similar odd extension of g. Let $\tilde{u}(x,t)$ denote the solution to the wave equation on the whole real line with initial data $\tilde{f}$ and $\tilde{g}$, so that

$$\tilde{u}(x,t) = \frac{1}{2}\left(\tilde{f}(x-ct) + \tilde{f}(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{g}(z)\,dz. \tag{5}$$

Now take $u(x,t) = \tilde{u}(x,t)$ for $x \geq 0$. The function u obviously satisfies the wave equation, since $\tilde{u}$ also does, and u has the right initial conditions. The only question is whether $u(0,t) = 0$ for all $t > 0$ (equivalently, $\tilde{u}(0,t) = 0$). From equation (5) we have

$$\tilde{u}(0,t) = \frac{1}{2}\left(\tilde{f}(-ct) + \tilde{f}(ct)\right) + \frac{1}{2c}\int_{-ct}^{ct} \tilde{g}(z)\,dz.$$

But $\tilde{f}(-ct) = -\tilde{f}(ct)$, so those terms cancel. Also, $\tilde{g}$ is odd, so the integral is zero; it works!

There's more to say, though. Here's where a picture is helpful:

For a point (x_0, t_0) which lies below the line $x = ct$ (i.e., $x_0 > ct_0$), as illustrated on the left, the function $\tilde{u}$ is synthesized from initial data from $x = x_0 - ct_0$ to $x = x_0 + ct_0$, and here $f = \tilde{f}$ and $g = \tilde{g}$. For such points we can use the standard D'Alembert formula (4), without reference to any odd extensions of f and g. This is merely another consequences of causality: such points lie too far from the endpoint $x = 0$ to be affected by the boundary by time $t = t_0$.

On the other hand, for a point (x_0, t_0) which lies above the line $x = ct$ (so $x_0 < ct_0$) we really do need to use (5). But that formula can be massaged into a slightly nicer form that also makes no reference to odd extensions. Refer to the above picture on the right: The integral of $\tilde{g}$ from $x = x_0 - ct_0$ to $x = x_0 + ct_0$ splits into two integrals, one from $x = x_0 - ct_0$ to $x = 0$, the other from $x = 0$ to $x = x_0 + ct_0$. We write $\tilde{u}(x_0, t_0)$ as

$$u(x_0,t_0) = \frac{1}{2}\left(\tilde{f}(x_0 - ct_0) + \tilde{f}(x_0 + ct_0)\right) + \frac{1}{2c}\int_{x_0 - ct_0}^{0} \tilde{g}(z)\,dz + \frac{1}{2c}\int_{0}^{x_0 + ct_0} \tilde{g}(z)\,dz$$

But $\tilde{f}(x_0 + ct_0) = f(x_0 + ct_0)$, and between $x = 0$ and $x = x_0 + ct_0$ in the last integral above we have $\tilde{g} = g$. We can thus write

$$u(x_0, t_0) = \frac{1}{2}\left(\tilde{f}(x_0 - ct_0) + f(x_0 + ct_0)\right) + \frac{1}{2c}\int_{x_0 - ct_0}^{0} \tilde{g}(z)\, dz + \frac{1}{2c}\int_{0}^{x_0 + ct_0} g(z)\, dz$$

Now make use of the fact that $\tilde{f}(z) = -f(-z)$ for $z < 0$ to write $\tilde{f}(x_0 - ct_0) = -f(ct_0 - x_0)$. Also, in the range $x_0 - ct_0 \leq z < 0$ we have $\tilde{g}(z) = -g(-z)$. We now have

$$u(x_0, t_0) = \frac{1}{2}\left(-f(ct_0 - x_0) + f(x_0 + ct_0)\right) - \frac{1}{2c}\int_{x_0 - ct_0}^{0} g(-z)\, dz + \frac{1}{2c}\int_{0}^{x_0 + ct_0} g(z)\, dz$$

One final change of variables in the first integral on the right above (substitute $-z \to z$, so $dz = \to -dz$) and lumping the resulting integral together gives

$$u(x, t) = \frac{1}{2}\left(f(ct + x) - f(ct - x)\right) + \frac{1}{2c}\int_{ct - x}^{ct + x} g(z)\, dz \qquad (6)$$

where I also dropped the subscripts on the variables. This gives the solution for $x < ct$.

Problem 1: Replace the boundary condition $u(0, t) = 0$ by the so-called *Neumann condition* $\frac{\partial u}{\partial x}(0, t) = 0$ for all t. Find a formula for the solution to the wave equation with the Neumann condition on the half-line analogous to (6) for $x < ct$. Hint: Make EVEN extensions of f and g to the whole line.

Solution on Bounded Intervals

The above trick for the half-line can also be used to write down the solution to the wave equation on a bounded interval. We won't do this in great detail, for there are better ways to do this problem, but it's kind of fun to think about (for a few minutes!) For simplicity take the interval to be $(0, 1)$. Suppose we want a solution to the wave equation for $0 < x < 1$ and $t > 0$ with initial data f and g, both defined for $0 \leq x \leq 1$, and we also want $u(0, t) = u(1, t) = 0$ at all times (the string is tied at both ends).

To solve this we can extend f and g periodically to the whole real line, as odd functions. Specifically, set $\tilde{f}(x) = -f(-x)$ for $-1 < x < 0$, and then extend $\tilde{f}$ to the whole real line as a function with period 2. Here's a crude picture:

Do the same for g to obtain $\tilde{g}$. The original D'Alembert formula (4) with $f = \tilde{g}, g = \tilde{g}$ yields the solution! One could even try to manipulate the formula to get it in terms of f and g alone, with no reference to the periodic extensions $\tilde{f}$ and $\tilde{g}$, but this turns out to be a bit of mess. You end up with infinitely many different formulas, depending on the point (x, t)! In any case, you ought to convince yourself (draw a picture) that $u(0, t) = u(1, t)$ at all times. It's very similar to the half-line case. So we've proved that the wave equation has a solution on a bounded interval!

Problem 2: Draw a picture and explain why $u(0, t) = u(1, t)$.