The Wave Equation II

MA 436
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1 Introduction

Recall that the wave equation in one dimension is

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

for $-\infty < x < \infty$ and $t > 0$. The initial conditions are

$$u(x,0) = f(x), \tag{2}$$
$$\frac{\partial u}{\partial t}(x,0) = g(x) \tag{3}$$

for some specified functions f and g. Equation (2) dictates the initial position of the string, while (3) is the initial velocity.

The three basic questions we ask about any PDE are

1. *Existence*: Is there a solution to the PDE with the given additional conditions (in our case, the wave equation with initial conditions (2) and (3))?

2. *Uniqueness*: Is there only one solution?

3. *Stability*: How sensitively does the solution depend on the initial conditions? Specifically, do small changes in the initial conditions result in small changes in the solution? (This has big implications for numerically solving the PDE).

Definition: Problems for which all three answers are "yes" are called *well-posed*. If a problem is not well-posed it's said to be *ill-posed*.

2 The D'Alembert Solution

The easiest way to solve the wave equation is via simple intuition: the solution ought to look like waves moving to the left or right at some speed k.

It's simple to check that if $\phi(z)$ is some function defined on the real line then $u_1(x,t) = \phi(x - kt)$ is a "wave" that moves to the right at speed k, while $u_2(x,t) = \phi(x + kt)$ is a "wave" that moves to the left at speed k. Plugging $u_1(x,t) = \phi(x - kt)$ into the wave equation forces

$$(k^2 - c^2)\phi''(x) = 0$$

from which we must conclude that $k = \pm c$, at least if $\phi''(x)$ isn't identically zero. The same conclusion follows from plugging u_2 into the wave equation. We conclude that for *any* twice-differentiable function $\phi(z)$ the functions $\phi(x - ct)$ and $\phi(x + ct)$ are both solutions to the wave equation. It's easy to check this directly too, by plugging these functions back into the wave equation.

Note that the wave equation is linear—from our point of view this is a HUGE asset, not just for the wave equation, but for any PDE. Linearity here let's us assert that any linear combination of two solutions is again a solution, so that anything of the form

$$u(x,t) = \phi_1(x - ct) + \phi_2(x + ct) \tag{4}$$

is a solution for any choice of ϕ_1 and ϕ_2, at least if ϕ_1 and ϕ_2 are both twice differentiable in x and t. It's also easy to see that any scalar multiple of a solution, e.g., $A\phi(x - ct)$ or $A\phi(x + ct)$, is again a solution.

We can build a solution to the wave equation by rigging up a linear combination of the form (4) for suitably chosen ϕ_1 and ϕ_2. Specifically, the initial conditions dictate that at time $t = 0$ we need both

$$\phi_1(x) + \phi_2(x) = f(x), \tag{5}$$

$$-c\phi_1'(x) + c\phi_2'(x) = g(x). \tag{6}$$

Use equation (5) to find $\phi_2 = f - \phi_1$ and substitute this into equation (6) to obtain

$$-2c\phi_1'(z) + cf'(z) = g(z)$$

where I've used z for the independent variable instead of x. Solve this equation for $\phi_1'(z)$ and integrate from $z = 0$ to $z = x$ to obtain

$$\phi_1(x) = \phi_1(0) + \frac{1}{2}f(x) - \frac{1}{2}f(0) - \frac{1}{2c}\int_0^x g(z)\,dz.$$

If you look at equation (5) you'll see that there is a certain amount of leeway in choosing ϕ_1 and ϕ_2—I can add some constant to ϕ_1 if I subtract it from ϕ_2.

With this trick I can arrange for $\phi_1(0) = \frac{1}{2}f(0)$ (and so also $\phi_2(0) = \frac{1}{2}f(0)$) and so the previous equation is just

$$\phi_1(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(z)\,dz. \tag{7}$$

The same procedure shows that

$$\phi_2(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(z)\,dz. \tag{8}$$

Using equations (7) and (8) in equation (4) shows that the solution to the wave equation with initial conditions (2) and (3) can be written "explicitly" as

$$\begin{aligned} u(x,t) &= \phi_1(x-ct) + \phi_2(x-ct), \\ &= \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) \\ &\quad -\frac{1}{2c}\int_0^{x-ct} g(z)\,dz + \frac{1}{2c}\int_0^{x+ct} g(z)\,dz \\ &= \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz. \end{aligned} \tag{9}$$

The last equality follows from the basic fact that $\int_a^b = -\int_b^a$.

3 Examples

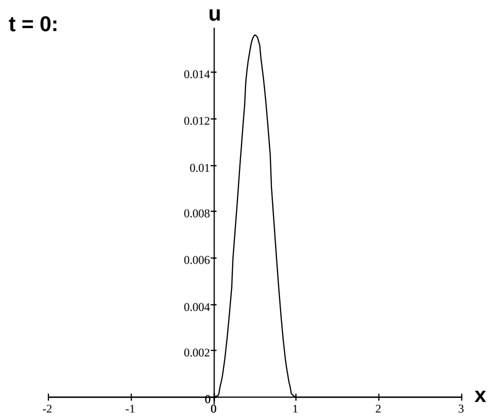
It appears that the functions f and g can be just about anything, although we need g to be nice enough so that the integral on the right in (9) exists. One way to do this is to require that g be continuous (piecewise continuous would be fine too). In fact, we'll put some more conditions on f and g shortly.

Here are some examples illustrating the nature of the solution.

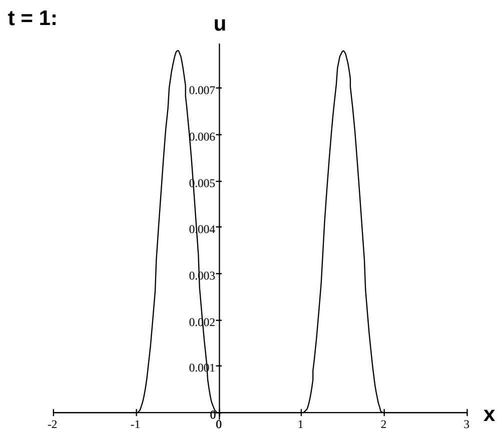
Example 1: Suppose that $c = 1$ and the initial conditions are

$$f(x) = \begin{cases} x^3(1-x)^3, & 0 < x < 1, \\ 0 & \text{else} \end{cases}$$

and $g(x) = 0$ for all x. The function $u(x,t)$ at time $t = 0$ looks like



According to the formula (9) the solution for $t > 0$ is $\frac{1}{2}(f(x+t) + f(x-t))$. The initial wave above splits into two waves, both initially equal to $f(x)/2$; one wave moves to the left at speed 1, the other to the right at speed 1. The solution at time $t = 1$ looks like



It's easy to see that this is always the behavior of the solution if $g = 0$.

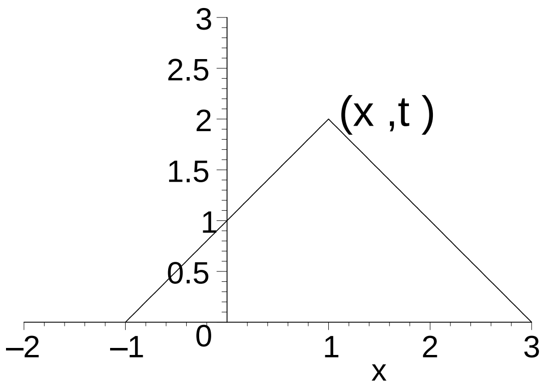
Example 2: This example, a bit more complicated, raises some issues about what it means for a function $u(x,t)$ to "satisfy" the wave equation. Let $c = 1$. Suppose now that $f \equiv 0$ and

$$g(x) = \begin{cases} 1, & 0 < x < 1, \\ 0 & \text{else} \end{cases}$$

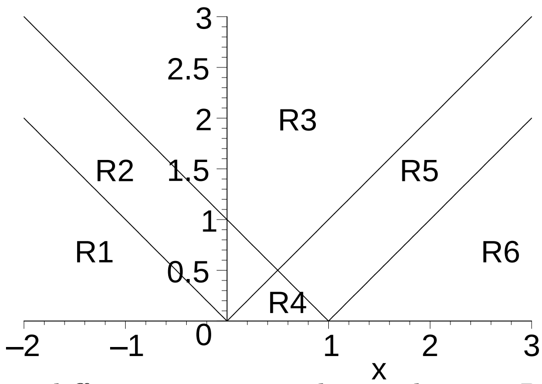
In this case the solution isn't so easy to write down explicitly, at least not without a bit of thought. We have

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} g(z)\,dz.$$

In order to figure out what the solution looks like it will be helpful to draw a picture in xt space:



Suppose we want to find the solution at some specific point (x_0, t_0). According to (9) we should integrate $g(z)$ (which "lives" at time $t = 0$) from $x = x_0 - t_0$ to $x = x_0 + t_0$, as illustrated in the picture above. I've drawn lines from the point (x_0, t_0) to $(x_0 - t_0, 0)$ and $(x_0 + t_0, 0)$. The triangular region between the lines is called the *backward light cone*. In this case if none of the light cone intercepts that portion of the x axis where $0 < x < 1$ then the integral $\int_{x_0-t_0}^{x_0+t_0} g(z)\,dz$ is zero and so $u(x_0, t_0) = 0$. You can easily check that this occurs if $x + t \leq 0$ or $x - t \geq 1$. In fact, there are six distinct cases, as illustrated by the figure below:



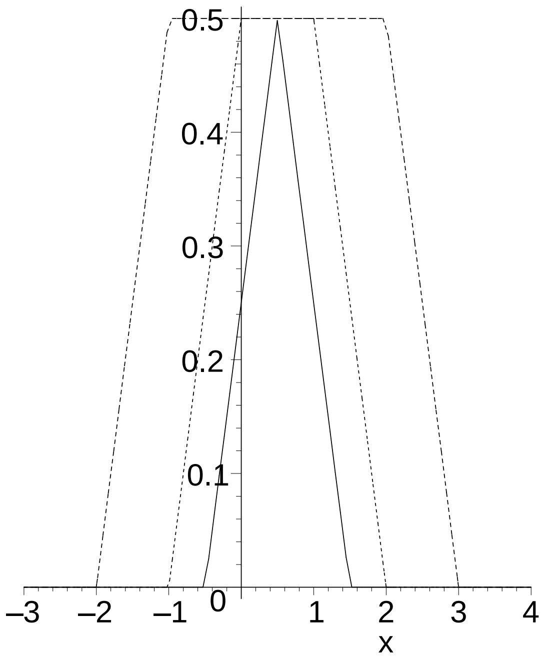
I've labelled the six different regions in the xt plane as R_1, R_2, ..., R_6. They are defined by the inequalities

$$\begin{aligned} R_1 &: \quad x+t \leq 0 \\ R_2 &: \quad x+t \leq 1, x+t \geq 0, x-t \geq 0 \\ R_3 &: \quad x-t \leq 0, x+t \geq 1 \\ R_4 &: \quad x-t \geq 0, x-t \leq 1 \\ R_5 &: \quad x-t \geq 0, x+t \leq 1, x-t \leq 1 \\ R_6 &: \quad x-t \geq 1 \end{aligned}$$

If you pick a typical point (x_0, t_0) in one of the regions, draw its light cone, and look at how much of g is intercepted, you find that the solution $u(x_0, t_0)$ is given in each region by

$$\begin{aligned} R_1 &: \quad u(x_0,t_0) = 0 \\ R_2 &: \quad u(x_0,t_0) = \tfrac{1}{2}(x_0+t_0) \\ R_3 &: \quad u(x_0,t_0) = \tfrac{1}{2} \\ R_4 &: \quad u(x_0,t_0) = \tfrac{1}{2}(1-x_0+t_0) \\ R_5 &: \quad u(x_0,t_0) = t_0 \\ R_6 &: \quad u(x_0,t_0) = 0 \end{aligned}$$

Here's a picture of the solution at times $t = 0.5, 1, 2$:



Here's an interesting point, though: The "solution" u is NOT even once differentiable at certain points, as you can see from above. What does it mean to say that this function "satisfies" the wave equation? We'll look at such issues later in the course, in a more general setting.

4 Existence and Uniqueness

Technically, in order for the solution (9) to make sense we need to be able to put it into the wave equation and get zero, which requires that we be able to differentiate u twice with respect to t and x. This requires that the initial condition f be twice differentiable and g must be once differentiable. As illustrated in Example 2 above, this isn't true unless f and g are themselves smooth enough. Hence we'll assume for now that f and g are smooth enough so that the second derivatives of u make sense; specifically, we'll require that f be twice continuously differentiable (that is, have a second derivative which is continuous) and g be once continuously differentiable.

Notation: We'll say that a function $\phi(x)$ is in $C^k(I)$ if ϕ is k times continuously differentiable on the interval I.

Thus in the case that f is in $C^2(\mathbb{R})$ and g is $C^1(\mathbb{R})$ we've answered the *existence* question for the wave equation, by showing that there is a solution to the wave equation with given initial conditions. But could there be more than one solution? The answer is NO, at least not a physically reasonable solution. The nicest way to prove this is by using conservation of energy. Before we show that the wave equation has a unique solution let's look at conservation of energy as it applies to this problem.

4.1 Conservation of Energy

We showed in a previous class that the total energy $E(t)$ (kinetic plus potential) of the string at time t as it vibrates is given by

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} (c^2 u_x^2(x,t) + u_t^2(t,x))\, dx, \tag{10}$$

where I've used the values $\lambda = 1$ and $T = c^2$.

Claim: $\frac{dE}{dt} = 0$, i.e., $E(t)$ is constant. This means that the total energy of the string is constant.

Proof: Differentiate both sides of (10) to obtain

$$\frac{dE}{dt} = \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} (c^2 u_x^2(x,t) + u_t^2(t,x))\, dx. \tag{11}$$

What I want to do is slip the t derivative inside the integral, but this merits some kind of remark. This is permissible only under certain conditions. What we are asking is in essence whether

$$\frac{d}{dt}\left(\int_a^b \phi(x,t)\, dx\right) = \int_a^b \frac{\partial \phi}{\partial t}(x,t)\, dx \tag{12}$$

for a function $\phi(x,t)$. The simplest circumstances under which this is true is the following:

1. Both ϕ and $\frac{\partial \phi}{\partial t}$ are continuous functions;
2. The interval (a,b) is finite, or equivalently, if the interval is infinite then ϕ is zero outside some finite interval).

In this case equation (12) is true (though it remains true under more general conditions).

Terminology: A function $\phi(x)$ which is identically zero for all x outside some bounded interval $[a, b]$ is said to have *compact support*.

What this means in our case is that we must assume that the function $u(x, t)$ has continuous second derivatives (we're using $\phi = u_x^2 + u_t^2$) and we must assume that for each fixed t the function $u(x, t)$ has compact support in x. We can then slip the derivative under the integral in equation (11) to obtain

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} (c^2 u_x u_{xt} + u_t u_{tt})\, dx \tag{13}$$

Now integrate the first term by parts in x (and use the fact that u is zero for $|x|$ large, so the endpoint terms vanish). We find

$$\int_{-\infty}^{\infty} c^2 u_x u_{xt}\, dx = -\int_{-\infty}^{\infty} c^2 u_{xx} u_t\, dx.$$

But if we use the fact that $c^2 u_{xx} = u_{tt}$ (u satisfies the wave equation) then

$$\int_{-\infty}^{\infty} c^2 u_x u_{xt}\, dx = -\int_{-\infty}^{\infty} c^2 u_{tt} u_t\, dx.$$

and equation (13) immediately becomes

$$\frac{dE}{dt} = 0$$

as claimed.

4.2 Uniqueness

Let's suppose that we have a solution to the wave equation (1) and initial conditions (2) and (3). We know that there is at least one solution to this problem, provided f and g are nice enough. I claim that there is *only* one solution (with the property that u has compact support in x for each fixed t). To prove this, suppose that there are TWO solutions to this problem, say $u_1(x, t)$ and $u_2(x, t)$, so both satisfy the wave equation and have the same initial conditions, and both have compact support in x for each time t. We will prove that $u_1(x,t) = u_2(x,t)$, i.e., they're really the same function. We'll start by defining a function $w(x,t) = u_2(x,t) - u_1(x,t)$; we'll then show that $w(x,t) \equiv 0$, so that $u_2 \equiv u_1$.

By using linearity it's easy to check that

$$\begin{aligned} \frac{\partial^2 w}{\partial t^2} - c^2 \frac{\partial^2 w}{\partial x^2} &= 0, \\ w(x,0) &= 0, \\ \frac{\partial w}{\partial t}(x,0) &= 0. \end{aligned}$$

We know that the total energy

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty} (w_t^2(x,t) + c^2 w_x^2(x,t))\, dx$$

is constant in time. But it's trivial to see that $E(0) = 0$, so $E(t) = 0$ for all t. This means that

$$\int_{-\infty}^{\infty} (w_t^2(x,t) + c^2 w_x^2(x,t))\, dx = 0$$

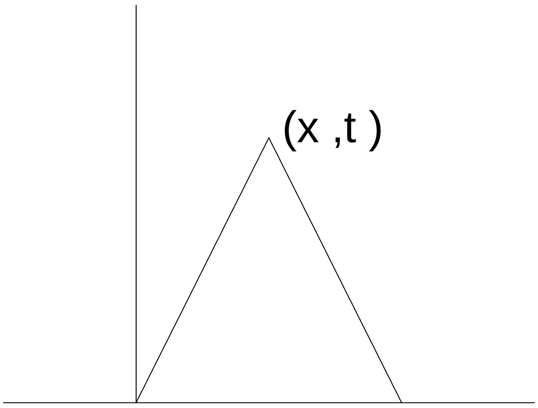
However, the integrand is a nonnegative function, and so the only way the integral can be zero is if $w_t^2(x,t) + c^2 w_x^2(x,t) \equiv 0$, so that both $w_t(x,t) \equiv 0$ and $w_x(x,t) \equiv 0$. This means that w is constant in both x and t, and since it starts as zero, it's always zero. Thus $w(x,t) \equiv 0$, so $u_1 \equiv u_2$, i.e., the wave equation has a unique solution. This is worth stating as a Theorem:

Theorem: Let $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$ have compact support. Then equation (9) defines the unique solution $u(x,t)$ to the wave equation with compact support in x for all $t > 0$ and initial conditions (2) and (3).

By the way, this argument requires that we consider only initial conditions which give $E(0) < \infty$, or else the integrals that appear don't make sense. The condition that $E(0) < \infty$ is simply that the solution has a finite amount of energy, the only case that makes sense physically. The condition $E(0) < \infty$ is actually guaranteed by our other assumptions (think about it: why?). It's also interesting to note that the integral which defines $E(t)$ and makes uniqueness so easy to prove would have been hard to come up with in a purely mathematical fashion—remember, we were led to $E(t)$ by thinking physically, in terms of energy.

4.3 Causality

Once again, let's consider the figure



The region inside the lines $x + ct = x_0 + ct_0$ and $x - ct = x_0 - ct_0$ is called the light cone. If you look at the D'Alembert solution

$$u(x_0, t_0) = \frac{1}{2}(f(x_0 - ct_0) + f(x_0 + ct_0)) + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} g(z)\, dz$$

you see that the solution $u(x_0, t_0)$ is synthesized out of data along the line $t = 0$ which lies between $x = x_0 - ct_0$ and $x = x_0 + ct_0$. Specifically, we integrate g from $x = x_0 - ct_0$ to $x = x_0 + ct_0$, and average the value of f at $x = x_0 - ct_0$ and $x = x_0 + ct_0$. This is the basis of the *Principle of Causality* for the wave equation. The initial conditions outside the backward light cone of the point (x_0, t_0) have no effect on the solution at (x_0, t_0). There's nothing special about initial conditions at $t = 0$, either; we could just as well take initial conditions at $t = 1$, in which we'd find that only the initial conditions on the plane $t = 1$ which lie inside the backward light cone are relevant to determining the solution at (x_0, t_0). The general principle is that only events that happen inside the light cone of (x_0, t_0) can affect what happens there.

The flip side of the coin is that events at the point (x_0, t_0) can influence the solution to the wave equation only at those points in the forward light cone. The Principle of Causality is sometimes stated "information cannot travel faster than c."

Editorial Remark: Causality is a perfect example of the kinds of interesting properties we look for when we try to understand PDE's and how solutions behave. Note that I said "understand PDE's" and not "solve PDE's". If you're hell bent on nuts-and-bolts techniques for cranking out solutions for PDE's, you'd probably never hit upon a phenomena like causality, and your understanding of what's really going on would be the poorer for it.

4.4 Stability

The goal in this section is to show that the solution to the wave equation is stable with respect to the initial conditions. In other words, small changes in the initial conditions produce small changes in the solution at any later time. If that were not the case then trying to solve the wave equation numerically would be difficult or hopeless—any numeric approximation in the initial conditions might change the solution at a later time by an arbitrarily large amount.

Notation: For a function $w(x)$ defined on the real line we're going to use the notation

$$\|w\|_\infty \equiv \sup_{-\infty<x<\infty} |w(x)|$$

and for a function $w(x,t)$ we'll define

$$\|w\|_{\infty,T} \equiv \sup_{-\infty<x<\infty} |w(x,T)|$$

for a fixed time T. You should easily be able to convince yourself that each of the following is true:

$$\begin{aligned}
\|w(x-a)\|_\infty &= \|w(x)\|_\infty, \text{ for any constant } a, \\
\|w_1 + w_2\|_\infty &\leq \|w_1\|_\infty + \|w_2\|_\infty, \\
\|w_1 - w_2\|_\infty &\leq \|w_1\|_\infty + \|w_2\|_\infty, \\
\left| \int_a^b w(x)\,dx \right| &\leq (b-a)\|w\|_\infty.
\end{aligned}$$

The quantity $\|w\|_\infty$ is called the *supremum norm* of the function w, and is one way to measure how large a function is.

Suppose that $u_1(x,t)$ solves the wave equation (with speed c) and initial conditions $u_1(x,0) = f_1(x)$ and $\frac{\partial u_1}{\partial t}(x,0) = g_1(x)$. Suppose that $u_2(x,t)$

solves the wave equation with initial conditions $u_2(x,0) = f_2(x)$ and $\frac{\partial u_2}{\partial t}(x,0) = g_2(x)$. Then the function $w(x,t) = u_2(x,t) - u_1(x,t)$ satisfies

$$\begin{aligned}
\frac{\partial^2 w}{\partial t^2} - c^2\frac{\partial^2 w}{\partial x^2} &= 0, \\
w(x,0) &= f_2(x) - f_1(x), \\
\frac{\partial w}{\partial t}(x,0) &= g_2(x) - g_1(x).
\end{aligned}$$

The D'Alembert solution to this is

$$\begin{aligned}
w(x,t) &= \frac{1}{2}((f_2(x-ct) - f_1(x-ct)) + (f_2(x+ct) - f_1(x+ct))) \\
&\quad + \frac{1}{2c}\int_{x-ct}^{x+ct} (g_2(z) - g_1(z))\, dz \qquad (14)
\end{aligned}$$

Let's fix $t = T$ (some specified future time T). Then

$$\begin{aligned}
\frac{1}{2}(f_2(x-cT) - f_1(x-cT)) &\leq \frac{1}{2}\|f_2 - f_1\|_\infty, \\
\frac{1}{2}(f_2(x+cT) - f_1(x+cT)) &\leq \frac{1}{2}\|f_2 - f_1\|_\infty, \\
\frac{1}{2c}\int_{x-cT}^{x+cT} (g_2(z) - g_1(z))\, dz &\leq T.\|g_2 - g_1\|_\infty
\end{aligned}$$

All in all we find from equation (14) that

$$|w(x,T)| \leq \|f_2 - f_1\|_\infty + T\|g_2 - g_1\|_\infty$$

or in terms of u_1 and u_2,

$$|u_2(x,T) - u_1(x,T)| \leq \|f_2 - f_1\|_\infty + T\|g_2 - g_1\|_\infty$$

This is valid for any x, so we can write it as

$$\|u_2 - u_1\|_{\infty,T} \leq \|f_2 - f_1\|_\infty + T\|g_2 - g_1\|_\infty$$

This is the stability result we want. It says that if f_1 is close to f_2 and g_1 is close to g_2 (as measured in supremum norm) then u_1 is close to u_2 at any fixed time T. Small changes in the initial conditions will produce small changes in the solution at any fixed future time. Of course as T increase the resulting changes might very well grow.

A typical application of this would be in a numerical solution to the wave equation, in which we might have to approximate the initial conditions—perhaps f_1 and g_1 are the "true" initial conditions and f_2 and g_2 are our numerically approximate conditions. Then u_1 is the true solution and u_2 is some kind of approximation. If we're interested the case time $T = 3$, for example, and if our solution must be accurate to 0.001, then we must have the initial conditions approximated accurately enough to guarantee

$$\|f_2 - f_1\|_\infty + 3\|g_2 - g_1\|_\infty \leq 0.001.$$

This puts an upper bound on how much error we can tolerate in our approximate initial conditions.

We can summarize most of what we've figured out concerning the wave equation with the following theorem:

Theorem: *The wave equation with initial conditions (2) and (3) is well-posed, i.e., possesses a unique solution which, for each $t > 0$, has compact support in x, and which depends stably on f and g, provided that $f \in C^2(\mathbb{R})$, $g \in C^1(\mathbb{R})$ are of compact support.*