The Wave Equation I

MA 436
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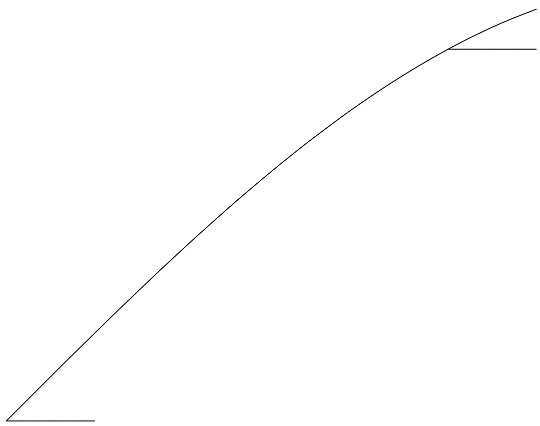
1 Introduction

Consider a string stretching along the x axis, of indeterminate (or even infinite!) length. We want to derive an equation which models the motion of the string as it vibrates. Let's use t for time, x for position along the horizontal axis, $u(x, t)$ for the vertical displacement of the string at position x and t. We'll also use T to denote the tension in the string and $\lambda(x)$ to denote the linear density of the string at position x.

The equation of motion of the string can be derived from nothing more than $F = Ma$ and a few reasonable assumptions. First, we will assume that the motion of the string is of small amplitude, and purely vertical. As such, the equation we derive won't apply to large amplitude motion or to a string with any significant longitudinal motion (e.g., a slinky). I'm going to use the notation u_x for the first derivative of $u(x, t)$ with respect to x, u_{xx} for the second derivative, etc.

Consider a small portion of the string stretching from x to $x + dx$, with angles θ_1 and θ_2 as labelled.



Our first task is to find the total force on this piece of the string. It's easy to check that the force on the left side of this small string element is $\mathbf{F}(x) = -T\cos(\theta_1)\mathbf{i} - T\sin(\theta_1)\mathbf{j}$. Similarly the force on the right end is $\mathbf{F}(x+dx) = T\cos(\theta_2)\mathbf{i} + T\sin(\theta_2)\mathbf{j}$ (this all assumes tension is roughly constant). Let us assume that the string vibrates in such a way that the slopes (or angles θ_1 and θ_2) stay small; this might be called a *small strain* or *first order* model, depending on your field. In this case both angles are close to zero and we have the approximations (from the Taylor's series)

$$\begin{aligned} \sin(\theta) &= \theta + O(\theta^3), \\ \cos(\theta) &= 1 - O(\theta^2). \end{aligned}$$

In this case we can make the first order approximations (dropping the small quadratic and higher terms)

$$\begin{aligned} \mathbf{F}(x) &\approx -T\mathbf{i} - T\theta_1\mathbf{j}, \\ \mathbf{F}(x+dx) &\approx T\mathbf{i} + T\theta_2\mathbf{j}. \end{aligned}$$

It's also easy to combine the above approximations to find that if θ is small then $\tan(\theta) \approx \theta$. From the picture it's clear that $u_x(x,t) = \tan(\theta_1) \approx \theta_1$ and $u_x(x+dx,t) = \tan(\theta_2) \approx \theta_2$, so to first order we have

$$\begin{aligned} \mathbf{F}(x) &\approx -T\mathbf{i} - Tu_x(x,t)\mathbf{j}, \\ \mathbf{F}(x+dx) &\approx T\mathbf{i} + Tu_x(x+dx,t)\mathbf{j}. \end{aligned}$$

The total force on this string element is $\mathbf{F}(x) + \mathbf{F}(x+dx)$, which is

$$\begin{aligned} \mathbf{F}_{tot} &= T(u_x(x+dx,t) - u_x(x,t))\mathbf{j} \\ &\approx Tu_{xx}(x,t)\,dx\,\mathbf{j}, \end{aligned} \tag{1}$$

where I've used the fact that $u_{xx}(x,t) \approx (u_x(x+dx,t) - u_x(x,t))/dx$ if dx is small—indeed, that is the very definition of the derivative u_{xx}. Note that the force is (to our level of approximation) entirely vertical, in keeping with our original assumptions.

The acceleration of this piece of the string is just $u_{tt}(x,t)\mathbf{j}$, and the mass is approximately $\lambda(x)\,dx$. Using Newton's second law, $F = ma$, we can equate the total force $\mathbf{F}_{tot}$ in equation (1) with $Ma = \lambda(x)u_{tt}(x,t)\,dx$ to find (I'll switch to Leibnitz notation for now)

$$\lambda(x)\frac{\partial^2 u}{\partial t^2} - T\frac{\partial^2 u}{\partial x^2} = 0, \tag{2}$$

which is one version of the *wave equation*. We can also write it in the form

$$\frac{\partial^2 u}{\partial t^2} - \frac{T}{\lambda(x)}\frac{\partial^2 u}{\partial x^2} = 0.$$

It's interesting to look at the physical dimensions of $\frac{T}{\lambda(x)}$; tension T has units of mass per length per time squared, while λ has units of mass per length. Thus $\frac{T}{\lambda(x)}$ has units of length squared per time squared, or velocity squared. That's exactly what it turns out to be. In the case that λ is constant (that's what we'll be most interested in) we frequently write $c^2 = T/\lambda$ and write the wave equation as

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0, \tag{3}$$

where c has dimensions of velocity.

Equation (3) is called the *wave* equation. It's an example of a *partial differential equation* ("PDE" for short), i.e., an equation involving the derivatives of an unknown function of two or more variables. The wave equation is one of the "big three" PDE's from mathematical physics (the other two are the heat equation and Laplace's equation).

Our goal is, of course, to find solutions to the wave equation. There are many, and we can't nail down one without further information. It seems obvious that we ought to need the initial position of the string in order to determine its position at later times and positions. Thus, if we take $t = 0$ as the initial time, we need the information that $u(x,0) = f(x)$ for some given function $f(x)$ that specifies the initial position of the string. Here x will range over the length of the string.

The initial position might seem like enough information to determine the string's motion, but it isn't. Do a simple thought experiment, in which two different (but physically identical) strings start in the same position, but with different initial velocities; it's clear the strings would have different future motion. So as it turns out, we also need to know the initial velocity of the string, say $\frac{\partial u}{\partial t}(x,0) = g(x)$ for some specified function $g(x)$.

For now we're going to concentrate on "infinite" strings, in which x takes the range $-\infty < x < \infty$. So in summary, out goal is to examine the solvability of the partial differential equation (3) with initial conditions

$$\begin{aligned} u(x,0) &= f(x) \\ \frac{\partial u}{\partial t}(x,0) &= g(x) \end{aligned}$$

for some given functions f and g.

2 Energy

For later reference, it's convenient to compute the kinetic and potential energy of the string as it moves. Again consider the small string element from x to $x+dx$. Its velocity is just $u_t(x,t)$ and its mass is (to first order) $\lambda(x)\,dx$. The kinetic energy of this piece is then $\frac{1}{2}\lambda(x)u_t(x,t)^2\,dx$ and the total kinetic energy of the string is obtained by adding up all the pieces, i.e.,

$$\text{KE} = \frac{1}{2}\int_a^b \lambda(x)u_t(x,t)^2\,dx \tag{4}$$

where a and b are the ends of the string (maybe at plus or minus infinity).

The potential energy of the string in a given configuration is a bit more challenging to find. Suppose that at some instant in time the string has the shape $u(x,t) = \phi(x)$; I'm thinking of time as frozen, since we're interested only in potential energy. The potential energy of the string in this position is the amount of work needed to deform it from the position $u(x,t) \equiv 0$, if we use the $u \equiv 0$ as our reference point. Consider deforming the string from this base position to $u(x,t) = \phi(x)$ by taking $r\phi(x)$ and letting r run from 0 to 1. For a given string element stretching from x to $x+dx$ the above analysis shows that the vertical force required to push the element upward for any value of r is $-Tr\phi_{xx}\,dx$ (this is MINUS the force exerted by the string.) If we change r by a small amount dr then we move this string element by a distance $\phi(x)\,dr$. The formula that work equals force times distance shows that we do an amount of work

$$dW = -T\phi(x)\phi_{xx}(x)r\,dx\,dr$$

on the string element. The total work on the element from $r = 0$ to $r = 1$ is obtain by adding (integrating) dW from $r = 0$ to $r = 1$, and is just

$$W_{element} = -\frac{T}{2}\phi(x)\phi_{xx}(x)\,dx.$$

The total work done in moving the entire string from the base configuration to the $u(x,t) = \phi(x)$ configuration is obtained by adding the work over each element, and is thus

$$W_{total} = -\frac{T}{2}\int_a^b \phi(x)\phi_{xx}(x)\,dx.$$

If we integrate this by parts in x (use $\int u\,dv = uv - \int v\,du$ with $u = \phi$, $dv = \phi_{xx}\,dx$) we obtain the potential energy of the string in the configuration $\phi(x)$:

$$\text{PE} = \frac{T}{2}\int_a^b \phi_x^2(x)\,dx$$

where in doing the integration by parts I've made the assumption that $u(a,t) = 0$ and $u(b,t) = 0$ OR $u_x(a,t) = 0$ and $u_x(b,t) = 0$; these boundary conditions model specific physical situations that we'll talk about later. All in all then, the potential energy of the string in the position $u(x,t)$ is

$$\text{PE} = \frac{T}{2}\int_a^b u_x^2(x,t)\,dx. \tag{5}$$

The total energy of the string, kinetic plus potential, at time t is

$$\text{Energy} = \frac{1}{2}\int_a^b (Tu_x^2(x,t) + \lambda(x)u_t^2(x,t))\,dx$$