

The Wave Equation

Mathematical Modelling Week 4

Kurt Bryan

As we've seen, solutions to the advection equation

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0$$

are of the form $u(x, t) = f(x - ct)$, waves moving to the right. It's easy to check that solutions to

$$\frac{\partial u}{\partial t} - c \frac{\partial u}{\partial x} = 0$$

are of the form $u(x, t) = f(x + ct)$, waves moving to the left. In many systems which display wave-like behavior, however, waves can move either to the left or to the right, or both. What kind of equation might govern such a situation?

To answer this we need to find a PDE for which functions of the form $u(x, t) = f(x - ct)$ and $v(x, t) = f(x + ct)$ are both solutions, simultaneously. To do this we first need the notion of a *differential operator*. A differential operator is something like

$$\frac{d}{dx}, \quad \frac{d^2}{dt^2}, \quad \frac{\partial}{\partial x}, \quad \text{or} \quad \frac{\partial^2}{\partial t \partial y}.$$

A differential operator acts on functions to produce new functions, much as a function acts on numbers to produce new numbers. For example, the differential operator d/dx acts on the function $f(x) = x^2$ to produce a new function $f'(x) = 2x$.

Differential operators can be added and subtracted in an obvious way; for example, the differential operator obtained by adding $\partial/\partial t$ and $\partial/\partial x$ satisfies

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) f(x, t) = \frac{\partial}{\partial t} f(x, t) + \frac{\partial}{\partial x} f(x, t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}.$$

Differential operators can also be composed, just like functions. Just apply one operator followed by the other:

$$\frac{\partial}{\partial t} \frac{\partial}{\partial x} f(x, t) = \frac{\partial}{\partial t} \left(\frac{\partial}{\partial x} f(x, t) \right) = \frac{\partial}{\partial t} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial t \partial x}$$

Exercise: Simplify the differential operator

$$\frac{\partial}{\partial t} \frac{\partial}{\partial x} - \frac{\partial}{\partial x} \frac{\partial}{\partial t}$$

Justify your answer.

A solution $u(x, t)$ to the right-moving advection equation is a “root” of the differential operator $\partial/\partial t + c\partial/\partial x$, i.e.,

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = 0,$$

while a solution $v(x, t)$ to the left-moving equation is a root of $\partial/\partial t - c\partial/\partial x$. We need to cook up an equation which is satisfied simultaneously by both u and v . Here is one inspiration for finding such an operator: Suppose you have a function $f(x) = x - 4$; obviously $x = 4$ is a root for this function. Suppose also that $g(x) = x + 2$; obviously $x = -2$ is a root for g . How can we find a function for which BOTH $x = -2$ and $x = 4$ are roots? By multiplying f and g ; define $h(x) = f(x)g(x) = (x - 4)(x + 2)$. Then both $x = 4$ and $x = -2$ are roots of h .

It's this simple example that leads us to try composing our differential operators, to construct a new PDE for which waves can move left or right. We try

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\phi(x, t) = \left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)\phi(x, t) = 0.$$

(where did the cross terms go?) or just

$$\left(\frac{\partial^2\phi}{\partial t^2} - c^2\frac{\partial^2\phi}{\partial x^2}\right) = 0. \tag{1}$$

This is called the *wave equation*. The constant c is the speed of propagation for the waves. We haven't really proved that the wave equation governs any particular physical system, only that it has waves moving left and right as solutions. Later we'll make some more compelling arguments that this equation really governs certain systems.

Exercises:

1. Verify that $\phi(x, t) = f(x - ct) + g(x + ct)$ is a solution to the wave equation, for any functions f and g .
2. If we want to solve the wave equation for $-\infty < x < \infty$ and $t > 0$ we need two initial conditions, usually of the form $\phi(x, 0) = p(x)$ (initial position) and $\phi_t(x, 0) = q(x)$ (initial velocity). It turns out that for any such initial data the solution can be written out as in problem (1), as a superposition of a left and right moving wave. Find f and g in terms of p and q .