# Unbounded Operators

MA 466
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Not all interesting linear operators on a Hilbert space are bounded—differentiation is the "prime" example (ha ha–get it?) Let $T$ be a linear operator with domain $D(T)$ where $D(T)$ is a subspace of a Hilbert space $H$. We'll also assume that $D(T)$ is dense in $H$, so that the closure of $D(T)$ is all of $H$. In this case $T$ is said to be *densely defined*. Of course $T$ could be bounded, but then it can be extended to a bounded operator on all of $H$, (by continuity: if $x \in H$ but $x$ isn't in $D(T)$, choose a sequence $x_n \subset D(T)$ which converges to $x$, set $T(x) = \lim_n T(x_n)$; it's well-defined). So we'll stick to the case in which $T$ is unbounded.

**Example 1:** Let $H = L^2(a,b)$ (with functions taking real values). Let $T$ denote differentiation with domain $D(T) = \{f \in H : f \in C^1[0,1]\}$. The set $D(T)$ is dense in $H$. Also, $T$ is unbounded, for the sequence $f_n = \sin(nx)$ is bounded in $L^2(a,b)$, but $T(f_n) = n\cos(nx)$ is unbounded.

One can define the adjoint operator for an unbounded densely-defined operator $T$. What we'd like that $T^*$ satisfy $< Tx, y >=< x, T^*y >$ for all $x \in D(T)$ and all $y \in D(T^*)$. Before we define $T^*$ we should figure out what to take for $D(T^*)$. In fact we'll choose $D(T^*)$ to be all $y \in H$ such that the linear functional $x \rightarrow < Tx, y >$ is continuous for $x \in D(T)$ (this doesn't say what $D(T^*)$ is very explicitly though). Since $D(T)$ is dense this allows us to extend the functional to all $x \in H$ as outline in the first paragraph. The Riesz representation then dictates the existence of a unique $g \in H$ such that $< Tx, y >=< x, g >$ for all $x \in H$. We define the adjoint as $T^*y = g$. This uniquely determines $T^*$ on its domain.

**Example 2:** This continues Example 1. Let $g$ be a $C^1[0,1]$ function (so $g \in L^2(0,1)$) with $g(0) = g(1) = 0$; let us use $V$ to denote this subspace of $L^2(0,1)$ (it's also dense). Then

$$< Tf, g >= \int_0^1 f'(x)g(x)\,dx.$$

Integrate by parts to find that

$$< Tf, g >= -\int_0^1 f(x)g'(x)\, dx.$$

If we take $T^*(f) = -f'$, then we have $< Tf, g >=< f, T^*g >$ for all $f \in C^1[0,1]$ and $g \in V$. The domain $V$ may not be as large as possible (that is, it may be possible to enlarge the class of $g$ for which we obtain $< Tf, g >=< f, T^*g >$), but in any case we've defined an adjoint for $T$ on a dense subspace of $H$.

An operator is said to be "symmetric" if $< Tx, y >=< x, Ty >$ for all $x, y \in D(T)$ (note that $x$ AND $y$ must both be in $D(T)$). If it's the case that $D(T^*) = D(T)$ then the operator is said to be self-adjoint.

**Example 3:** This is a small variation on Example 2. Let $H = L^2(0,1)$ where the functions may take values in $\mathbb{C}$. Let $T$ be the operator $T(f) = if'$ with domain $D(T) = \{f \in H : f \in C^1[0,1], f(0) = f(1)\}$. The adjoint of $T$ is defined much as in Example 2: Let $g$ be a $C^1[0,1]$ function (so $g \in L^2(0,1)$) with $g(0) = g(1)$ (same conditions as $f$). Then

$$< Tf, g >= \int_0^1 if'(x)\overline{g(x)}\, dx.$$

Integrate by parts to find that

$$< Tf, g >= -i\int_0^1 f(x)\overline{g'(x)}\, dx =< f, Tg > .$$

In other words, $T$ is self-adjoint.

It turns out that one can define the spectrum of an unbounded operator (similar to bounded: if $(T - \lambda I)$ is injective on $D(T)$ and onto $H$ with bounded inverse, then $\lambda$ is said to be in the resolvent set of $T$; the spectrum $\sigma(T)$ consists of all $\lambda$ NOT in the resolvent set). The spectrum of an unbounded operator is always closed, but may not be bounded, and MAY be empty.

In Example 3, with $T(f) = if'$ with the requirement $f(0) = f(1)$, it's easy to see that the functions $\phi_n(x) = \cos((\pi/2 + 2\pi n)x)$ are eigenfunctions with eigenvalues $\lambda_n = \pi/2 + 2\pi n$ (which tend to infinity).