Markov Chains

Introduction

Consider a fixed population in which, at a certain initial time, 10 percent of the population lives in a rural setting, 60 percent live in the suburbs, and 30 percent live in an urban area. We can represent the initial state of the populace as a vector $\mathbf{x}_0$,

$$\mathbf{x}_0 = \begin{bmatrix} 0.1 \\ 0.6 \\ 0.3 \end{bmatrix}.$$

Let's refer to rural as "state 1", suburban as "state 2", and urban as "state 3", so each member of the population exists in one of these three states. Suppose also that each year exactly 10 percent of the people who live in a rural setting move to the suburbs, 5 percent move to an urban area, and the remaining 85 percent stay in a rural setting. Suppose also that each year exactly 2 percent of the people who live in a suburban setting move to a rural area, 25 percent move to an urban area, and the remaining 73 percent stay in the suburbs. Finally, suppose that each year exactly 1 percent of the people who live in an urban setting move to a rural area, 25 percent move to a suburban area, and the remaining 74 percent stay in an urban setting.

At the end of the first year, what does the population distribution look like? Well, we started with a fraction 0.1 of rural people, of which $(0.1)(0.85) = 0.085$ stayed rural. Of the suburban dwellers (0.6 fraction at the start of the year) we have a fraction 0.02 moving to the suburbs, a total fraction $(0.6)(0.02) = 0.012$ of the population. Finally, of the 0.3 fraction of urban dwellers, $(0.3)(0.01) = 0.003$ end up in a rural area. All in all at the end of the first year the fraction of the population living in a rural area will be

$$(0.1)(0.85) + (0.6)(0.02) + (0.3)(0.01) = 0.1$$

the same as at the start of the year.

Similar computations show that the fraction of people living in the suburbs at the END of the first year is

$$(0.1)(0.1) + (0.6)(0.73) + (0.3)(0.25) = 0.523$$

and the fraction of people living in urban areas at the END of the first year is

$$(0.1)(0.05) + (0.6)(0.25) + (0.3)(0.74) = 0.377.$$

Let $\mathbf{x}_1$ denote the population distribution at the end of the first year, so

$$\mathbf{x}_1 = \begin{bmatrix} 0.1 \\ 0.523 \\ 0.377 \end{bmatrix}.$$

A very convenient way to arrange the computations above is to simply note that $\mathbf{x}_1$ was computed as $\mathbf{x}_1 = \mathbf{T}\mathbf{x}_0$ where

$$\mathbf{T} = \begin{bmatrix} 0.85 & 0.02 & 0.01 \\ 0.1 & 0.73 & 0.25 \\ 0.05 & 0.25 & 0.74 \end{bmatrix}.$$

This is a model in which there are three states. The vector $\mathbf{x}_0$ is called the *initial state vector* and the matrix $\mathbf{T}$ is called the *transition matrix*. The matrix $\mathbf{T}$ quantifies how the state vector changes from one iteration of the model to the next. In this case each iteration is one year.

It's not hard to see that the population distribution at the end of the second year will be $\mathbf{x}_2 = \mathbf{T}\mathbf{x}_1 = \mathbf{T}^2\mathbf{x}_0$. More generally, the population distribution at the end of the nth year will be $\mathbf{x}_n = \mathbf{T}^n\mathbf{x}_0$.

Exercises

1. Note that the columns of $\mathbf{T}$ add up to one. Why?
2. What happens to $\mathbf{T}^n$ and $\mathbf{x}_n$ as n gets large? Does the population distribution approach a steady-state? What is it?

The General Case

Consider some "population" in which each individual can exist in any one of m different states (mutually exclusive). That means that at any given time a certain fraction x_k of the population exists in state number k. The vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

summarizes the fraction of the population that exists in each state. We'll use $\mathbf{x}_0$ for the initial state of the population. Suppose that at each iteration of the model a fraction T_{ji} of the individuals in state i move to state j. Then at iteration $n+1$ the population state is related to that at iteration n by the equation

$$\mathbf{x}_{n+1} = \mathbf{T}\mathbf{x}_n$$

where

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1m} \\ T_{21} & T_{22} & \cdots & T_{2m} \\ & & \vdots & \\ T_{n1} & T_{n2} & \cdots & T_{nm} \end{bmatrix}$$

As in exercise 1 above, the columns of this matrix must sum to one.

These kinds of models are an example of *Markov Chains*. Of particular interest in most situations is the long-term behavior of the state vector $\mathbf{x}_n$. It can be shown that under certain reasonable conditions the quantity $\mathbf{T}^n\mathbf{x}_0$ must converge to a vector $\mathbf{b}$ which is a steady-state, i.e., $\mathbf{T}\mathbf{b} = \mathbf{b}$, so $\mathbf{b}$ represents a population distribution which never changes.

Exercise

3. Markov chains are often used to model epidemics. Suppose that each member of a population exists in one of three states. State 1 (which we'll denote S for "susceptible") is the state in which an individual has never had a certain disease and can catch it. State 2 (which we'll denote I for "infected") is the state in which an individual actively has the disease. State 3 (which we'll denote R for "recovered") is the state in which an individual has recovered from the disease and is permanently immune. This well-known model is called the "SIR" model for epidemics.

   Suppose the transition fractions are $T_{11} = 0.9$, $T_{21} = 0.1$, $T_{31} = 0.0$, $T_{12} = 0.0$, $T_{22} = 0.05$, $T_{32} = 0.95$, $T_{13} = 0.0$, $T_{23} = 0.0$, and $T_{33} = 1.0$.

   (a) Write out the transition matrix $\mathbf{T}$. Why do all of the zero transition fractions above make sense?

   (b) In a Markov model a state is called *absorbing* if an individual can never leave that state. Which state in this model is absorbing?

   (c) Start with any initial state vector $\mathbf{x}_0$ you like (but make sure its entries add to one). Compute $\mathbf{T}^{100}\mathbf{x}_0$. What happens? Interpret it in plain English—does it make sense?

   (d) Modify the transition matrix so that who have recovered have a 0.05 chance of re-infection (but can't become susceptible—that state is reserved for those who have never been infected). Re-compute $\mathbf{T}^{100}\mathbf{x}_0$. Any difference? Interpret!

4. (Extra Credit) To the SIR model above (with no reinfection possible), add a fourth state: "dead". Suppose that $T_{41} = 0$, $T_{42} = 0.07$, $T_{43} = 0.0$, and of course $T_{44} = 1.0$. Also assume that of the infected people, 5 percent will remain infected, 88 percent will be recovered. Write out the transition matrix $\mathbf{T}$. Pick an initial state vector corresponding to a completely healthy populace and compute $\mathbf{T}^n\mathbf{x}_0$ for large n. Interpret the results. What percentage of the population will ultimately die as a result of the disease?