Traffic Flow II

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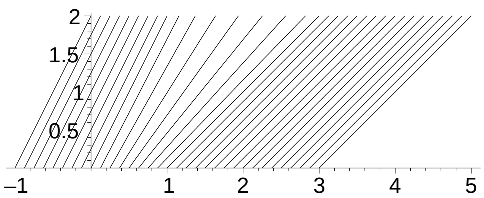
Quick Recap

Last time we were modelling the flow of traffic on a freeway. Conservation of cars (no on or off ramps) on the free implies that $\rho_t + q_x = 0$ holds. We also have $q = c\rho$ where c is the speed at which cars are moving, and we allow c to depend on ρ as $c = 1 - \rho$ (in units where the max speed is 1 and max traffic density is 1). We thus had $q = \rho - \rho^2$ and the continuity equation yielded the traffic equation

$$\rho_t + (1 - 2\rho)\rho_x = 0 \tag{1}$$

with initial condition $\rho(x, 0) = \phi(x)$.

The characteristics for this equation have constant slope that depends on the solution (in particular, they are dictated by the initial condition). The characteristics typically look like

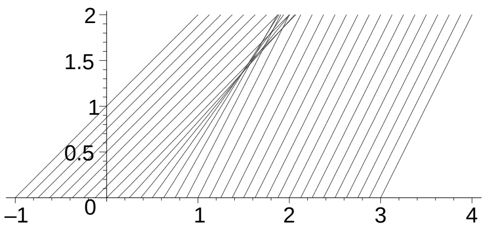


To compute $\rho(x, t)$ at any point we find the characteristic it's on, follow it back to initial point $(x_0, 0)$, and find $\rho(x, t) = \phi(x_0)$.

But sometimes things aren't so nice! Consider the traffic equation with initial condition

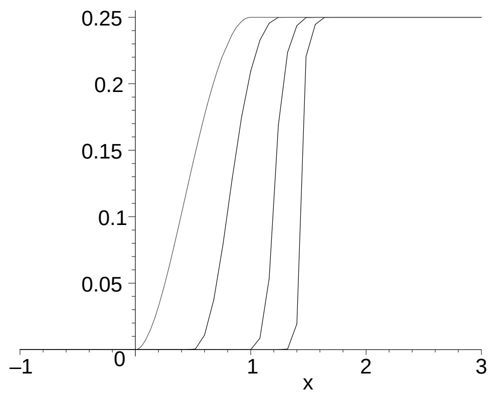
$$\phi(x) = \begin{cases} 0, & x < -1 \\ \frac{1}{4}(x^2(2-x)^2), & -1 \leq x \leq 0 \\ \frac{1}{4}, & x > 1 \end{cases}$$

This models the case in which there is relatively slow moving dense traffic for $x > 1$, very light faster moving traffic coming up from behind for $x < 0$, and a transition region of slowing down traffic for $0 < x < 1$. For this initial data the characteristics look like



Big trouble! For large enough t you can see points through which many different characteristics pass—which solution value should we assign here?

Note that the trouble starts around $t = 1.3$, roughly. Let's look at the actual solution at times $t = 0.0, 0.5, 1.0, 1.3$ to see what's happening. They're graphed below; as t increases the steep region moves to the right.



The solution gets steeper and steeper as we approach $t = 1.3$. Physically, the faster moving cars in back are catching up to the slower cars in front. The density profile becomes arbitrarily steep at about time $t = 1.3$. At this point a so-called *shock wave* or just *shock* develops. The terminology is from gas/fluid dynamics, where this type of phenomena was first encountered.

After the shock develops, the "solution" to the PDE becomes multi-valued, and the density goes past vertical—it's not even a function anymore. At this point either the model or our solution procedure breaks down. What should we do?

Solutions

One approach to resolving the problem is to modify our model, that is, change the PDE. Another approach is to keep the PDE but modify our solution procedure, and even modify what we mean by "solution". Let's first consider changing the model.

There are lots of ways to modify the model to prevent the formation of shocks. One approach is to let the traffic speed c depend not only on ρ, but also on ρ_x. The idea is that drivers choose their speed based not only on the local traffic density, but also on whether density is increasing or decreasing up ahead, and how rapidly. This could be accounted for by replacing the equation $c = 1 - \rho$ with

$$c = 1 - \rho - \lambda \rho_x \tag{2}$$

for some small parameter λ.

Exercises

- What should the sign of λ be? Why?

With this modification the traffic equation becomes

$$\rho_t + (1 - 2\rho - \lambda \rho_x)\rho_x - \lambda \rho \rho_{xx} = 0. \tag{3}$$

The hope is that the λ terms should counteract the development of shocks; as drivers approach a steep region the slow down drastically, and the density doesn't develop a vertical slope.

Proving that the modification really prevents shocks from developing is hard, and we won't mess with it. There's at least one drawback to this model too: it's now second order in x, and it's not clear what the characteristics are, or even whether they exist. Try finding them! More generally, it's not obvious how to solve this PDE, although later we'll look at simple numerical schemes approximating the solution.

A slight variation on this modification involves focusing on changing $q(x,t)$ directly, instead of c. Instead of taking $q = \rho - \rho^2$ we take $q = \rho - \rho^2 - \lambda \rho_x$. The idea is to decrease the flow-rate directly when a steeply increasing gradient is encountered.

Exercise

- Work out the new traffic equation using constitutive relation $q = \rho - \rho^2 - \lambda \rho_x$.

This equation too prevents the formation of shocks. And it turns out to be solvable in a reasonably closed-form way, but it's a bit of work. We'll talk about that later.

Shocks

There is another approach to dealing with the crossing of characteristics and infinite slope of the solution ρ. We modify what we mean by a "solution" to the PDE. Essentially, we let the solution slope become vertical at some instant in time. At this moment the solution has a jump discontinuity and hence is no longer differentiable. Thus it can't satisfy the PDE in any classical sense. What we need to do is make sense of discontinuous "solutions" to the PDE. The key is to go back to the conservation principle from which we derived the PDE, to introduce this notion of a travelling shock in a way that is faithful to the conservation physics of the situation. The situation is a lot like when we introduced the idea of Dirac "delta" functions in the DE course: They're a mathematical tool that allows us to more easily model and deal with situations involving impulsive and discontinuous phenomena.

Consider the following situation: We have a traffic density $\rho(x,t)$ which for times $t < t_1$ is differentiable and satisfies the traffic PDE. For times $t \geq t_1$ a shock has developed. The shock moves with position $x = x_s(t)$. At this point the solution ρ has a jump discontinuity: it limits to different (finite) values from the left and right side of the shock. At all points away from the shock $\rho(x,t)$ is a classical smooth solution to the PDE. As a result, q is also smooth away from the shock.

The first thing we need to figure out is how the shock moves. Consider an observer which is just to the left of the shock following it at the same speed, moving along a curve $x = a(t)$, and another observer just to the right of shock, moving along curve $x = b(t)$ at the same speed as the shock. Thus $a(t) < x_s(t) < b(t)$, with $b(t) - a(t) \approx 0$. We'll use the interval $(a(t), b(t))$ as a moving control volume, and eventually consider the limit as $b - a \to 0$. Let $\rho^-(t)$ denote the limiting value of the density to the left of the shock and $\rho^+(t)$ the value at the right of shock, at time t, so that if we let $a(t)$ approach $x_s(t)$ from the left we have $\rho(a(t),t) \to \rho^-(t)$. Similarly $\rho(b(t),t) \to \rho^+(t)$ as $b(t)$ approaches $x_s(t)$ from the right.

The rate at which cars enter the control volume at $x = a(t)$ is $q(a(t),t) - \frac{da}{dt}\rho(a(t),t)$. Think about this! The rate at which cars leave the control volume at $x = b(t)$ is given by $-q(b(t),t) + \frac{db}{dt}\rho(b(t),t)$. Overall, the rate R at which the amount of cars in the moving control volume change is given by the rate in plus the rate out, namely $-(q(b,t) - q(a,t)) + \rho(b(t),t)\frac{db}{dt} - \rho(a(t),t)\frac{da}{dt}$, or

$$R = -[q] + [\rho]\frac{dx_s}{dt} \tag{4}$$

where $[q] = q(b(t),t) - q(a(t),t) = q^+ - q^-$, $[\rho] = \rho(b(t),t) - \rho(a(t),t) = \rho^+ - \rho^-$, and I've used $\frac{da}{dt} = \frac{db}{dt} = \frac{dx_s}{dt}$.

I claim that $R = 0$. The heuristic argument is that since $b(t) - a(t)$ can be made arbitrarily small, there are no cars in the control volume, and so the rate of change is always zero. This is pretty lame, though—just because a quantity is small that DOES NOT mean its derivative is small.

To be more precise, we can count the rate at which cars in the control volume change in another way, namely as

$$R = \frac{d}{dt}\int_{a(t)}^{b(t)} \rho(x,t)\,dx.$$

We can work this integral out more explicitly, but we have to be careful: ρ has a jump discontinuity at $x = x_s$, so I'll split the integral into two pieces and simplify.

$$\begin{aligned}
R = \frac{d}{dt}\int_{a(t)}^{b(t)} \rho(x,t)\,dx &= \frac{d}{dt}\int_{a(t)}^{x_s(t)} \rho(x,t)\,dx + \frac{d}{dt}\int_{x_s(t)}^{b(t)} \rho(x,t)\,dx & (5)\\
&= \rho(x_s^-(t),t) - \rho(a(t),t) + \rho(b(t),t) - \rho(x_s^+(t),t) \\
&\quad + \int_{a(t)}^{x_s(t)} \rho_t(x,t)\,dx + \int_{x_s(t)}^{b(t)} \rho_t(x,t)\,dx & (6)\\
&= \rho(x_s^-(t),t) - \rho(a(t),t) + \rho(b(t),t) - \rho(x_s^+(t),t) \\
&\quad + \int_{a(t)}^{x_s(t)} q_x(x,t)\,dx + \int_{x_s(t)}^{b(t)} q_x(x,t)\,dx & (7)\\
&= \rho(x_s^-(t),t) - \rho(a(t),t) + \rho(b(t),t) - \rho(x_s^+(t),t) \\
&\quad + q(x_s^-(t),t) - q(a(t),t) + q(b(t),t) - q(x_s^+(t),t). & (8)
\end{aligned}$$

where $\rho(x_s^-(t),t) = \lim_{y\to x_s^-}(\rho(y,t))$ and $\rho(x_s^+(t),t) = \lim_{y\to x_s^+}(\rho(y,t))$. In going from (5) to (6) I used the basic rules for differentiating an integral with respect to its limits. For (6) to (7) I used $\rho_t = q_x$, and the for (8) I just applied the Fundamental Theorem of Calculus.

Now in equation (8) let $a(t) \to x_s(t)$ from below, $b(t) \to x_s(t)$ from the right. Since ρ and q are assumed continuous on either side of the shock, the whole thing goes to zero! This shows that $R = 0$, and so

$$-[q] + [\rho]\frac{dx_s}{dt} = 0.$$

This can be solved for $\frac{dx_s}{dt}$ as

$$\frac{dx_s}{dt} = \frac{[q]}{[\rho]}. \quad (9)$$

This is called the *Rankine-Hugoniot* formula for the shock speed.

For our traffic model we had $q = \rho - \rho^2$, so $[q] = [\rho] - [\rho^2]$ and we can substitute this into equation (9) to find that shocks must move at speeds which obey

$$\frac{dx_s}{dt} = 1 - \frac{[\rho^2]}{[\rho]}.$$

Exercise

- Give an example to show that $[\rho]^2 = [\rho^2]$ is completely false.

However, since $[\rho] = \rho^+ - \rho^-$, it's easy to work out that $[\rho^2] = (\rho^+ + \rho-)(\rho^+ - \rho^-)$ and so for the traffic situation

$$dx_s/dt = 1 - (\rho^- + \rho^+). \tag{10}$$

So the shock speed for the traffic equation is determined by the density value on either side of the shock.

Exercises

- Consider a solution to the traffic equation with initial condition

$$\phi(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

  Show that $\rho(x,t) = \phi(x)$ for all $t > 0$ satisfies the traffic equation at all points away from $x = 0$, and that the shock at $x = x_s(t) = 0$ satisfies condition (10). Interpret what this all means physically (what are the cars doing?)

- Consider a solution to the traffic equation with initial condition

$$\phi(x) = \begin{cases} 1/2, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

  The solution here has a shock starting at time $t = 0$ as position $x = 0$. The shock moves in a straight line. Find the equation $x_s(t)$ for the shock position (use (10) to find its speed). Write out the solution in closed-form. Hint: it's constant on either side of the shock. Again, interpret what this all means physically (what are the cars doing?)

- The characteristic through (x,t) for the traffic equation is found by solving $x_0 - 2t\phi(x_0) = x - t$ for x_0. If, for a fixed t and all x there's only one solution for x_0, then a shock cannot have developed by time t (no characteristics have yet crossed). In short, if we define $\psi(x_0) = x_0 - 2t\phi(x_0)$ then a "no shock" condition is that $\psi(x_0) = c$ has a unique solution for all c. This would be the case if $\psi'(x_0) > 0$ for all x_0. What does this last condition translate to in terms of ϕ and t. Give a lower bound (in terms of ϕ') for how long the solution will go without developing shocks.

A Few More Remarks

The last problem (and the method of characteristics) show that

Theorem: If the initial data $\phi(x)$ is differentiable then the traffic equation has a unique classical (no shock) solution for all $t < t_0$, where t_0 depends on ϕ.

The Theory of Shocks let's us push the solution farther in time, by extending what we mean by "solution" to the PDE. But there's a lot more to say about shocks. First, the Rankine-Huguniot condition (along with the PDE) doesn't actually nail down the solution uniquely (if the solution has shocks). For example, for the traffic equation with initial data

$$\phi(x) = \begin{cases} 1, & x < 0 \\ 0, & x \geq 0 \end{cases}$$

we can write down (at least) two distinct solutions. One solution is simply $\rho(x,t) = \phi(x)$, with a shock propagating up the t axis. Another is

$$\rho(x,t) = \begin{cases} 1, & x < -t \\ \frac{t-x}{2t}, & -t \leq x \leq t \\ 0, & x > t \end{cases}$$

This last solution has no shocks.

In order to obtain uniqueness of solutions, we need an additional condition, and this condition is derived (like the shock condition) from the physics of the situation. We won't go into them here, but they really on the concept of *entropy* and are called "entropy conditions". See Peter Lax's book "Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves" for more information (it's only 40 pages long!)