Traffic Flow I
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Introduction

Consider a stretch of freeway with no entrance or exit ramps. On such a stretch of freeway we clearly have "conservation of cars", barring any wrecks. Thus in our conservation framework we will today take the conduit to be such a freeway. The function $\rho(x,t)$ measures the traffic density of cars at some point x on the freeway at time t (maybe in units of cars per mile) while $q(x,t)$ measures how many cars per unit of time pass a given point on the road at time t. Note that cars are discrete objects, so at a sufficiently small scale the notion of "traffic density" breaks down. So assume we're dealing on a large enough length scale that we can sensibly think of cars as a continuously divisible quantity.

Since cars are conserved we must have the continuity equation

$$\rho_t + q_x = 0. \tag{1}$$

As with advection we need a constitutive relation between ρ and q.

The Constitutive Relation

Let c denote the speed at which the cars move, but we will NOT assume that c is constant. It depends on a lot of things, but we'll try today to capture the dependence of c on one important factor: the traffic density ρ itself. Specifically, if $\rho \approx 0$ then traffic moves along at some maximum speed, say c_0 (what assumptions are we making here?) On the other hand there is some maximum possible physical value for ρ, say M, bumper to bumper, and at this condition speed drops to roughly zero. A formula that captures these observations is

$$c(\rho) = c_0(1 - \rho/M). \tag{2}$$

This function just interpolates linearly between the light traffic and bumper to bumper extremes. Of course it's also dimensionally correct!

Now a bit of thought shows that just as with ordinary advection, the relation between ρ and q is $q = c\rho$. But since c depends on ρ according to equation (2) we find that the constitutive relation between ρ and q should be

$$q = c_0(1 - \rho/M)\rho. \tag{3}$$

For simplicity let's rescale length and time so that $c_0 = 1$ and $M = 1$ (go ahead and work out the rescaling). In this case we obtain just

$$q = (1 - \rho)\rho. \tag{4}$$

The Traffic Equation

Use equation (4) to replace q_x in the continuity equation. A bit of simplification shows that ρ satisfies

$$\rho_t + (1 - 2\rho)\rho_x = 0 \tag{5}$$

the so-called *traffic equation*. It looks a lot like the advection equation, except $1 - 2\rho$ has replaced the advection speed c. It seems reasonable that this equation should have a solution if we specify the traffic density at some initial time, say $t = 0$, as $\rho(x, 0) = \phi(x)$.

Equation (5) is a nonlinear PDE, and these are usually a prescription for pain. There is little likelihood of a closed-form solution. With a nonlinear PDE the best you can usually hope for (aside from solving it numerically) is to find a few simple special cases with closed form solution, maybe prove something about the general behavior of solutions (e.g., they exist, they're differentiable, or non-negative, etc.). In this case though there is some beautiful geometry lurking about that let's us write out closed form solutions for any initial data, more or less.

The Characteristics

The key to solving the traffic equation is to look for it's characteristics, that is, curves in the xt plane on which solutions to (5) are constant. Let's specify such a curve by giving x as a function of t, so $x = X(t)$ for some function X. If ρ is constant on such a curve then $\rho(X(t), t) = \text{const}$ for all t and by the chain rule

$$\frac{d}{dt}(\rho(X(t), t)) = \rho_t + \frac{dX}{dt}\rho_x = 0. \tag{6}$$

Now compare equation (6) and the traffic equation (5). If $x = X(t)$ is a curve on which ρ is constant then

$$\frac{dX}{dt} = 1 - 2\rho(X(t), t). \tag{7}$$

Unlike the advection case, the characteristics here will depend on the solution itself.

Now we seem to be in a Catch-22 situation here: If we knew the characteristics we could solve the PDE. However, in order to find the characteristics we need to know the solution to the PDE. We can't win!

But in fact the solution is right in front of us—it's so slippery that you'll need to think it through several times. Look at equation (7). On any characteristic curve $x = X(t)$ the value of ρ is constant. But this means (by equation (7)) that $\frac{dX}{dt}$ is also constant along the characteristic—the characteristic must therefore have a constant slope, and so the *characteristics are straight lines*. But their slopes are not all the same, and in fact the slope of the characteristic through a point (x_0, t_0) is $1 - 2\rho(x_0, t_0)$. Note this is the slope with t as run and x as rise. If you want the slope with x as run and t as rise (as we've conventionally been drawing our pictures) you should take $1/(1 - 2\rho)$. This observation let's us write out solutions (almost) explicitly.

Here's an example. Let the initial condition in the traffic equation be given by $\rho(x, 0) = \phi(x)$ with

$$
\phi(x) = \begin{cases} \frac{1}{4}, & x < 0 \\ \frac{1}{4}(1 - x^2)^2, & x < 1 \\ 0, & x \geq 1 \end{cases}
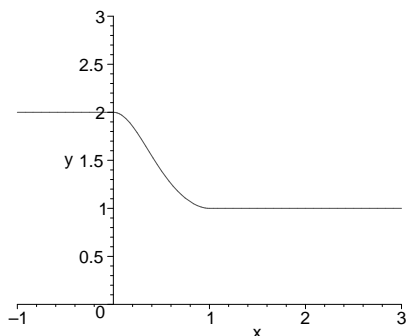$$

The function ϕ looks like
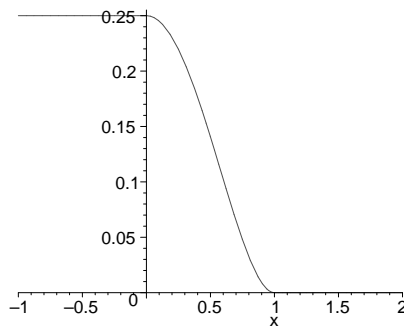
This represents heavier traffic to the left of $x = 0$ (but only at $1/4$ of the max density) and lighter traffic to the right.

Consider the characteristic curve passing through an initial point $(x_0, 0)$. Here $\rho = \phi(x_0)$, and so the characteristic curve through $(x_0, 0)$ has slope $m = 1/(1 - 2\phi(x_0))$, where x is run and t is rise. Here's a plot of the quantity
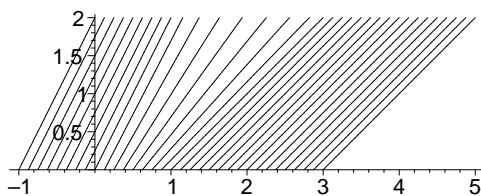
$1/(1 - 2\phi(x))$:

for our choice of ϕ.

For any given initial point $(x_0, 0)$ we can work out the characteristic curve through $(x_0, 0)$; it's a line with slope $1/(1 - 2\phi(x_0))$, and is thus given by (use the point-slope formula)

$$t = \frac{x - x_0}{1 - 2\phi(x_0)}. \tag{8}$$

Here is a plot of the characteristic curves for this choice of ϕ, through a variety of initial points.

On each such curve the solution to (5) is constant.

This gives a procedure for computing the solution at any given point (x, t): Plug the values for x and t into equation (8) and solve for x_0. This gives the initial point whose characteristic passes through (x, t). The value of $\rho(x, t)$ is then just $\phi(x_0)$. Here's a plot of the solution at time $t = 4$, along with a plot of the initial condition.

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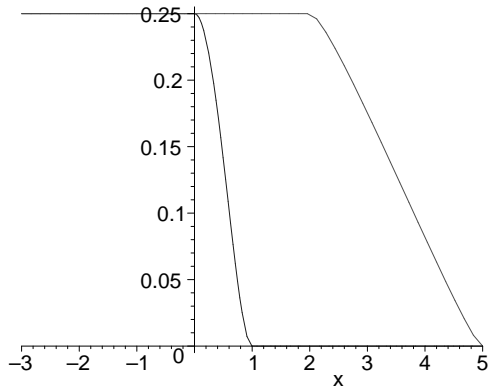
You should think this through to make sure you believe it's reasonable (or if not, why you think it's unreasonable!) It's a bit subtle to interpret what's going on. Recall that the actual speed of the cars is given by equation (2), or simply $c = 1 - \rho$ in our case. Here's a plot of the speed of the cars at time $t = 4$, as a function of their x coordinate:
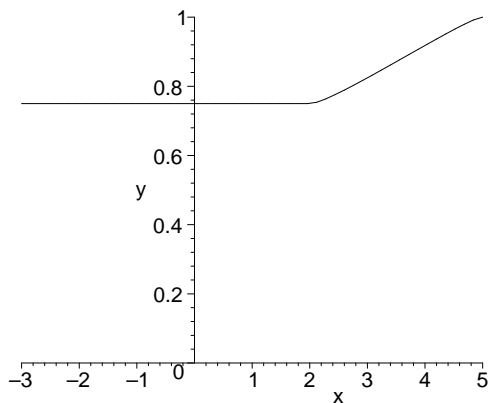
The cars to the left of about $x = 2$ are still stuck in the heavier traffic, and moving at speed 0.75. To the right of $x = 2$ traffic becomes progressively lighter and the speeds higher. As cars reach the transition region around $x = 2$ they begin to accelerate away from the trailing traffic. The slope of the solution in this transition region will gradually broaden with time.

One final note: Suppose that the initial condition $\phi(x)$ has compact support, i.e., $\phi(x)$ is non-zero only inside some finite interval (a, b) (note this

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was not that case in the examples above). The quantity $\int_{-\infty}^{\infty} \phi(x)\,dx$ is the total number of cars on the road, and one of our original assumptions what that this quantity is conserved. You can check this directly from the traffic equation. Specifically,

$$
\begin{aligned}
\frac{d}{dt}\left(\int_{-\infty}^{\infty} \rho(x,t)\,dx\right) &= \int_{-\infty}^{\infty} \rho_t(x,t)\,dx \\
&= -\int_{-\infty}^{\infty} (1 - 2\rho(x,t))\rho_x(x,t)\,dx \\
&= -\lim_{x_1\to-\infty, x_2\to\infty} (\rho(x,t) - \rho^2(x,t))|_{x=x_1}^{x=x_2} \\
&= 0
\end{aligned}
$$

where I've slipped $\frac{d}{dt}$ under the integral (permissible if ρ is smooth enough) and used the fact that $\rho - \rho^2$ is an anti-derivative for $(1 - 2\rho)\rho_x$. Since $\frac{d}{dt}(\int_{-\infty}^{\infty} \rho(x,t)\,dx) = 0$, this quantity is constant over time. The final equation itself dictates that whatever quantity ρ defines density for, that quantity is conserved.

Finally, there is one complication that can arise that I haven't mentioned. For nonlinear PDE's like the traffic equation it's possible (likely) that more than one characteristic will pass through a given point (x,t) in the plane, at least it t is large enough. This signals the beginning of so-called *shocks* in the solution. The procedure and equation above breaks down in this case. That's a topic for next week!