

Strategies for Problem Solving

Kurt Bryan

Below are some strategies to try when solving a difficult problem. These strategies are especially useful in steps 1 and 2 of problem solving—when you're trying to understand the problem and formulate an approach. Of course, they aren't guaranteed to work, but then, nothing is! Some I've already mentioned, but we'll go over them again. In each case I give an example to show how the strategy works.

1. **Play/Experiment/Guess/Look for Patterns.** This is one of the first things to try when confronted with a problem. For example, consider this problem.

Find a formula for $1^3 + 2^3 + 3^3 + \cdots + n^3$ in terms of n .

The first thing to do here is to stick in some specific values of n and see what happens:

$$\begin{aligned}1^3 &= 1, \\1^3 + 2^3 &= 9, \\1^3 + 2^3 + 3^3 &= 36, \\1^3 + 2^3 + 3^3 + 4^3 &= 100.\end{aligned}$$

Look at the pattern—the right sides are always perfect squares, 1^2 , 3^2 , 6^2 , 10^2 , etc. Also notice that $1 = 1$, $3 = 2 + 1$, $6 = 3 + 2 + 1$, $10 = 4 + 3 + 2 + 1$, and so on. We can guess that the $1^3 + 2^3 + 3^3 + 4^3 + 5^3 = (5 + 4 + 3 + 2 + 1)^2$, which is correct. From here you can probably guess

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.$$

In fact, you can take it a step further. The right side simplifies even more. I leave that to you.

2. **Draw a Picture.** I've already mentioned this one. It's usually obvious advice when you're dealing with a geometric problem (like the triangle problem), or in many physical problems. If you're not yet convinced of the value of pictures, consider this easy problem:

John starts from a given point and walks 2 miles north. Bill starts from the same points and walks 2 miles southeast. How far apart are Bill and John?

Try solving the problem without drawing a picture, even in your head! The usefulness of picture drawing isn't limited to geometric problems. For example, consider the classic problem:

Given two containers, one which holds 3 liters and one which holds 5 liters, and another large container filled with water, measure out exactly 4 liters of water.

For this problem it would make a lot of sense to draw a picture of a 3 gallon and 5 gallon container. In fact, you'll probably need to draw a succession of such pictures in order to solve the problem.

Actually, this problem is an example of another strategy that is sometimes useful:

- Work Backwards.** If the problem is such that a final answer or result is given and you're supposed to figure how to get there from here, working backwards sometimes helps. It's like solving one of those maze problems on paper; it might be easier to start at the end, or even to work toward the middle from the beginning and end simultaneously. For the 3 and 5 liter container problem above, we want to end up with 4 liters (obviously in the 5 liter container):

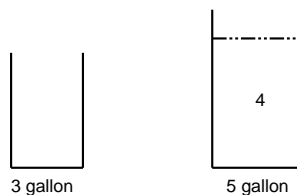


Figure 1: Solution to the puzzle.

What configuration would have preceded this one? If you think about it, it was probably

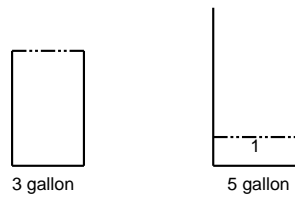


Figure 2: Previous step.

that is, a full 3 liter container and 1 liter in the 5 liter container. And before that, the 5 liter container was empty and the single liter must have been in the 3 liter container:

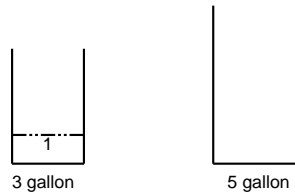


Figure 3: Another step back.

So we've worked back from the solution to an intermediate stage. Maybe you can now work from the start (both containers empty) forward to this stage—meet in the middle. Figure out how to get 1 liter into the 3 liter container and the problem will be solved.

4. **Introduce Suitable Notation.** Again, I've mentioned it before but it's so important I'll mention it again. Most problems involve equations relating different quantities. If you don't name the quantities involved, then you can't write equations and you can't solve the problem. An example (posed by the Indian mathematician Mahavira, and stolen from Pat Swickard's web page).

The mixed price of 9 citrons and 7 fragrant wood apples is 107, again, the mixed price of 7 citrons and 9 fragrant wood apples is 101. O you arithmetician, tell me quickly the price of a citron and of a wood apple here, having distinctly separated those prices well.

Look at the unknowns! We're supposed to find the price of a citron and the price of a wood apple, but we're going nowhere until we give those quantities symbolic names. How about C for the price of a citron and W for a wood apple? The first piece of information we're given is that

$$9C + 7W = 107.$$

You can probably figure it out from here, using the already mentioned fact that it takes n equations to find n unknowns.

5. **Simplify/Specialize/Generalize.** This is one of the most useful approaches to a really hard problem. Modify the problem to make it easier! Then solve the easier problem and maybe this will point the way to the solution of the original hard problem. For example, if there are several conditions given, try dropping one or more of them; if there is some unspecified variable in the problem, give it a specific value; if the problem involves large numbers or large collection of objects try substituting in a smaller number or collection; if the problem is three (or higher) dimensional, replace with a two-dimensional version. These usually makes the problem easier. For example

Find all triplets (x, y, z) of positive integers such that $xyz = x + y + z$.

Looks hard! How about trying a simpler problem first? We could try finding all pairs of positive integers (x, y) such that $xy = x + y$. That looks easier and might provide some inspiration. (I'll let you work it out). Or, going back to the original problem, we could simply set $z = 1$ and try solving $xy = x + y + 1$ (that's the original with $z = 1$). Then try $z = 2$, $z = 3$, etc. In short, reduce the original problem in three variables to a series of problems in two variables. Maybe you can think of other simplifications.

6. **Analogy.** Have you ever seen a problem like the one under consideration? If so, maybe you can use the solution to that problem to solve the one at hand. Or maybe the solution for the familiar problem will suggest an approach to the present problem. For example,

- Look at the problem above (3 integers whose sum equals their product). You have recently seen a very similar problem involving

three integers. Maybe the solution to that problem will yield the solution to the one above.

For more routine homework problems the examples in the book are usually similar to the problems assigned as homework. Understand and emulate them.

7. **Test by Dimension.** You've probably noticed and/or been told that all physical formulas balance dimensionally—the units must be the same on both sides of an equation. For example, the area A of a circle is given by $A = \pi r^2$, where r is the radius. Notice that area A comes in units of length squared, e.g., square meters, and r comes in units of length, so r^2 has units length squared. The number π is *dimensionless*; it has no units. Thus $A = \pi r^2$ balances dimensionally, since both sides have dimension length squared. This principle can be exploited to check whether a physical formula, or your answer to a physics problem, is correct. If it doesn't balance, it's wrong; if it does, it might be right (but there's no guarantee). An example:

An object is dropped from a height of h meters. If gravitational acceleration is g meters per second per second, how long does the object take to hit the ground?

Suppose, after some work, I come up with the answer that the time required is $\sqrt{2h/g}$ seconds. Could this be right? If it is correct then it should have units of seconds. The units on g are meters per second squared ($\frac{m}{s^2}$) and the units on h are meters (m). The constant 2 is dimensionless. Therefore $2g/h$ has units

$$\sqrt{\frac{m}{\frac{m}{s^2}}} = \sqrt{s^2} = s$$

so my formula is plausible. Of course, it could still be wrong—maybe it should be $\sqrt{5h/g}$. The dimensional analysis won't detect that kind of error.

8. **Go back to the definitions.** When a professor defines one or more new technical terms, he or she almost always gives a few problems that force you to review the terminology. For example, let's say you just learned the definition of a perfect number: A positive integer is *perfect* if the sum of its proper divisors equals the number. Here's a problem.

- *Find a perfect number which is divisible by 7.*

In order to solve this you actually need to know two definitions: “perfect number” and “proper divisor”. You may recall that an integer a is a proper divisor of another integer b if b is divisible by a and $a \neq b$. One way to solve the problem is to experiment—start checking multiples of 7 to see if they’re perfect. The only proper divisor of 7 is 1, but $7 \neq 1$, so 7 isn’t perfect. The proper divisors of 14 are 1, 2, 7, but $14 \neq 1 + 2 + 7$, so 14 isn’t perfect. You can figure out the rest of it!

9. **Persistence.** Finally, don’t underrate the value of persistence. Andrew Wiles worked almost exclusively for seven years on the proof of Fermat’s last theorem (that $x^n + y^n = z^n$ has no solutions in which x, y , and z are integers if $n \geq 3$). If you want a problem on which persistence will help, try proving (or disproving) Goldbach’s conjecture: every even number bigger than 2 can be written (in at least one way) as a sum of two prime numbers, e.g., $18 = 5 + 13$. Despite its simplicity, it remains unsolved after 250 years.