An Inverse Problem in Non-Destructive Testing

The "Reciprocity Gap" Approach
MA 436
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Introduction

Let D be a bounded domain in $\mathbb{R}^2$; think of D as some material object. At time $t = 0$ the object D has temperature $f(x)$, where $x = (x_1, x_2)$. We apply a heat flux $g(x, t)$ to the boundary of D, denoted by ∂D; recall that g measures the rate at which we pump heat energy into D, per unit time per length of the boundary. The resulting temperature $u(x, t)$ of D obeys the heat equation

$$\frac{\partial u}{\partial t} - \triangle u = 0$$

for $x \in D$ and $t > 0$, with initial condition $u(x, 0) = f(x)$ and Neumann boundary condition $\frac{\partial u}{\partial \mathbf{n}}(x, t) = g(x, t)$ for $x \in \partial D$ and $t > 0$. Here I've assumed for simplicity that all physical parameters (e.g., thermal conductivity and diffusivity) equal 1. It's also worth recalling that the quantity $\mathbf{q} = -\nabla_x u$ is the direction of the heat energy flow, where $\nabla_x u =< \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2} >$. The heat equation with this boundary and initial condition has a unique solution.

Now imagine that the object D has developed an internal "crack". This crack will be assumed to be a line segment σ contained in D, as illustrated below (D is the "blob"):

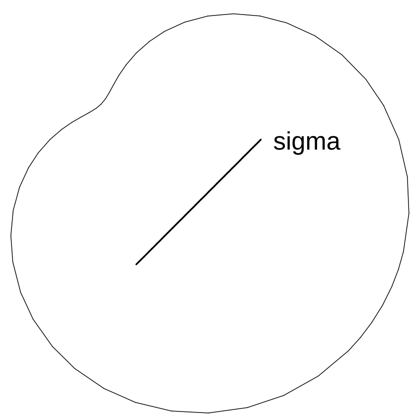

Figure 1

The presence of σ will alter the flow of heat through D—the solution u won't be same now, and in particular u on ∂D will change due to the presence of σ. Our goal is to use this fact to get an image of the crack σ, but without cutting open D or otherwise damaging it. Think of what follows as an actual physical experiment we can do (because it is!)

Specifically, we'll do this: start D with a known initial temperature, which for simplicity we'll take to be zero, then pump in a known heat flux g. We then measure the resulting temperature u on ∂D, since the outer boundary is the only part of D we can access non-destructively.

But how does σ affect the flow of heat energy? The simplest assumption is that σ completely blocks the flow of heat. If $\mathbf{n}$ denotes a unit normal vector field on σ then we model this as $-\nabla u \cdot \mathbf{n} = 0$ on both sides of σ, or equivalently, $\frac{\partial u}{\partial \mathbf{n}} = 0$ on σ.

All in all then, our goal is this: Given a domain D with an unknown internal linear crack σ, we apply input heat flux g to ∂D. The resulting temperature $u(x,t)$ satisfies

$$\frac{\partial u}{\partial t} - \triangle u = 0 \text{ in } D \setminus \sigma \quad (1)$$

$$\frac{\partial u}{\partial \mathbf{n}} = g \text{ on } \partial D \quad (2)$$

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \sigma \quad (3)$$

$$u(x,0) = 0 \quad (4)$$

where $\mathbf{n}$ is a unit normal outward vector field on ∂D or a unit normal vector on σ, as appropriate. We don't solve the above PDE (we don't know σ), but rather let Nature solve it, by doing the actual experiment. We then MEASURE $u(x,t)$ for $x \in \partial D$ and all $t > 0$ (or maybe on just a finite time interval $t \in (0,T)$). From this information we attempt to determine σ.

This is an example of an *inverse problem.* The normal setting for a PDE is that we're given the domain, the PDE, the boundary and initial conditions, and then we're supposed to find the solution to the PDE. An inverse problem is when we have the solution (or some portion thereof) and have to find one of the other quantities—an unknown initial condition, and unknown boundary condition, or in the present case, an unknown boundary. That's what σ is here—a kind of internal boundary of the cracked domain D.

The Time-Independent Case

Let's consider the case in which the problem is time independent—if the input flux g doesn't depend on time and if $\int_{\partial D} g\,ds = 0$ (net heat input is zero) then the solution $u(x,t)$ will quickly settle down to a steady-state configuration. Dropping the t derivatives in the above heat equation, as well as the initial condition gives

$$\triangle u = 0 \text{ in } D \setminus \sigma \tag{5}$$

$$\frac{\partial u}{\partial \mathbf{n}} = g \text{ on } \partial D \tag{6}$$

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \sigma \tag{7}$$

Condition (7) is to hold on BOTH sides of σ. The goal now: Given the input flux g and measurements of $u(x)$ for $x \in \partial D$, deduce σ. By the way, equations (5)-(7) also govern electrical conduction, where g is an input current flux applied to ∂D and u is the resulting electrical potential.

There's one small problem—equations (5)-(7) have a unique solution only up to an arbitrary additive constant (in the thermal interpretation, we need to specify our "zero degree" point; in the electrical interpretation we need to specify the "zero potential" point, i.e., the ground). We'll eliminate the lack of uniqueness by adding the condition $\int_{\partial D} u\,ds = 0$.

Note that the function u is required to be harmonic only on $D \setminus \sigma$, not all of D. The latter would require $u \in C^2(D)$, but the crack prevents this. Think of σ as an internal boundary of the region $D \setminus \sigma$, so u should not actually be harmonic ON σ, as we've discussed. A typical solution u looks like

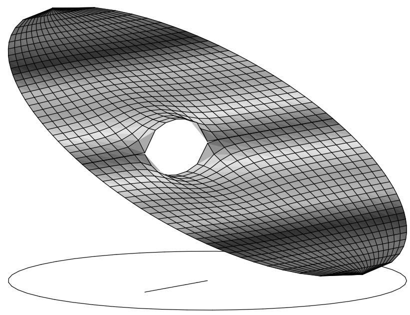

Figure 2

The bottom circle with the contained line segment is $D\backslash\sigma$; the surface floating above is the graph of $u(x_1, x_2)$. It might seem surprising that u should have a discontinuity across σ, but think of it like this: If I picked two points in D that are far apart, you wouldn't be surprised that the temperature at these points is quite different, right? Well points which lie just on opposite side of σ are in fact quite far apart, from the point of view of the heat flow—to get from one side of σ to the other you have to go all the way around the end, and so two such points may have quite different temperatures.

To aid in what follows, let's denote one side of the crack as the "plus" side and the other as the "minus" side. Let the unit normal vector $\mathbf{n}$ point from the minus to the plus side as illustrated below:

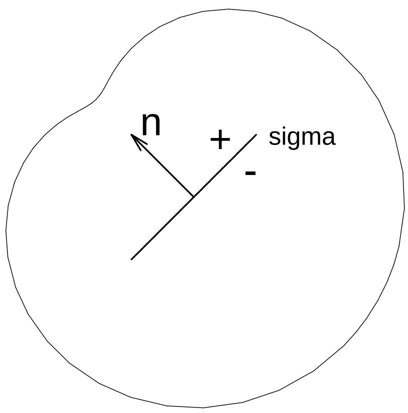

Figure 3

We'll add a "+" superscript to any function to denote that function's limiting value as its argument approaches σ from the plus side, and similarly a "−" superscript for the limiting values from the minus side. Thus for any point $x \in \sigma$ we can talk about $u^+(x)$ and $u^-(x)$ for the relevant limiting value of u as we approach x from either side of σ. Note from the previous discussion we don't expect $u^-(x) = u^+(x)$. In fact, let us define the quantity

$$[u](x) = u^+(x) - u^-(x)$$

for $x \in \sigma$. The function $[u](x)$ is called the "jump" of u over the crack σ.

The Inverse Problem Solution and Reciprocity Gap Formula

The reciprocity gap approach to finding the linear crack is based on Green's second identity, which states that if u and v are both harmonic in some region D then

$$\int_{\partial D} \left(v \frac{\partial u}{\partial \mathbf{n}} - u \frac{\partial v}{\partial \mathbf{n}} \right) ds = 0 \tag{8}$$

where $\mathbf{n}$ is an OUTWARD pointing unit normal vector on the boundary ∂D and ds is arc length.

Let $v(x)$ be a function which satisfies $\triangle v = 0$ in D (including on σ). Note that we can choose infinitely many such functions, and any function which is harmonic in D will be continuous and have continuous derivatives of all orders through D. Now think of approximating the linear crack σ with a thin region R which has non-zero width (but we'll let the width shrink to zero momentarily). Take ν as an outward unit normal vector field on $D \setminus R$ and note that ν points INTO R. Also, $\nu = \mathbf{n}$ on ∂D. Refer to the figure below:

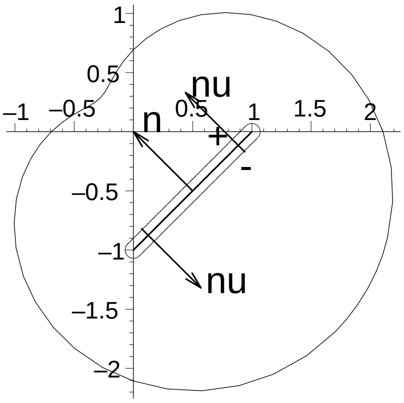

Figure 4

From Green's second identity we have (since u and v are both harmonic in $D \setminus R$; note D in equation (8) is replaced by $D \setminus R$ here, and ∂D is replaced with $\partial D \cup \partial R$)

$$\int_{\partial D} \left(v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right) ds + \int_{\partial R} \left(v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right) ds = 0. \tag{9}$$

Now let R shrink in width to approach σ. Of course this doesn't change the integral on ∂D. On ∂R however, we find that on the "minus" side of σ we have $\frac{\partial u}{\partial \nu} \to \frac{\partial u^-}{\partial \mathbf{n}}$, while on the "plus" side we have $\frac{\partial u}{\partial \nu} \to -\frac{\partial u^+}{\partial \mathbf{n}}$; refer to the figure above. Also, $u \to u^+$ on the plus side and $u \to u^-$ on the minus side. Note that all of these assertions require that the function and its derivatives are continuous up to σ (they are.) The integral over ∂R in equation (9) approaches

$$\int_\sigma \left(v^- \frac{\partial u^-}{\partial \mathbf{n}} - v^+ \frac{\partial u^+}{\partial \mathbf{n}} \right) ds - \int_\sigma \left(u^- \frac{\partial v^-}{\partial \mathbf{n}} - u^+ \frac{\partial v^+}{\partial \mathbf{n}} \right) ds.$$

But since $v^+ = v^-$ and $\frac{\partial v^+}{\partial \mathbf{n}} = \frac{\partial v^-}{\partial \mathbf{n}}$ (remember, v is smooth in D) and $\frac{\partial u^+}{\partial \mathbf{n}} = \frac{\partial u^-}{\partial \mathbf{n}} = 0$ the previous displayed expression is just

$$\int_\sigma [u] \frac{\partial v}{\partial \mathbf{n}} \, ds.$$

Substituting this into the ∂R integral in equation (9) and re-arranging gives the reciprocity gap formula

$$\int_\sigma [u]\frac{\partial v}{\partial \mathbf{n}}\,ds = \int_{\partial D}(u\frac{\partial v}{\partial \mathbf{n}} - v\frac{\partial u}{\partial \mathbf{n}})\,ds \tag{10}$$

where ds is arc length. Note that if we specify v on D then the entire right hand side of equation (10) is known. In fact, let us define the "reciprocity gap functional"

$$RG(v) := \int_{\partial D}(u\frac{\partial v}{\partial \mathbf{n}} - v\frac{\partial u}{\partial \mathbf{n}})\,ds. \tag{11}$$

We can compute $RG(v)$ for any chosen harmonic function v, because we know both u and $\frac{\partial u}{\partial \mathbf{n}}$ on ∂D. Equation (10) then becomes

$$\int_\sigma [u]\frac{\partial v}{\partial \mathbf{n}}\,ds = RG(v). \tag{12}$$

The above equation relates the unknown quantity σ to a quantity that we CAN compute, namely the right side. The trick is to use the ability to compute $RG(v)$ for any harmonic v to extract information about σ.

Identifying the Crack Angle

So far nothing we've done makes use of the fact that the crack is linear. Now we'll use that assumption. Let the crack be linear as illustrated in the previous figures, and oriented at some angle θ with respect to the horizontal axis. We can assume that $-\pi/2 < \theta \leq \pi/2$. You can easily check that a unit normal vector on σ is given by $\mathbf{n} = (-\sin(\theta), \cos(\theta))$, and of course this doesn't vary with position on σ. Note that we might as well assume that $0 \leq \theta < \pi$.

We'll make several different choices for the harmonic "test function" v, in order to extract information about σ from the boundary data. First, let $v_1(x_1, x_2) = x_1$ and use this for v in equation (12) (note that $\nabla v_1 = (1, 0)$, so $\frac{\partial v_1}{\partial n} = -\sin(\theta)$ on σ), to obtain

$$-\sin(\theta)\int_\sigma [u]\,ds = \int_{\partial D}(u\frac{\partial v_1}{\partial \mathbf{n}} - v_1\frac{\partial u}{\partial \mathbf{n}})\,ds = RG(x_1). \tag{13}$$

Now let $v_2(x_1, x_2) = x_2$ and use this for v in equation (12) (note that $\nabla v_2 = (0, 1)$, so $\frac{\partial v_2}{\partial n} = \cos(\theta)$ on σ), to obtain

$$\cos(\theta)\int_\sigma [u]\,ds = \int_{\partial D}(u\frac{\partial v_2}{\partial \mathbf{n}} - v_2\frac{\partial u}{\partial \mathbf{n}})\,ds = RG(x_2). \tag{14}$$

The right sides of equations (13) and (14) are known (or at least computable) quantities, since we specified $\frac{\partial u}{\partial \mathbf{n}} = g$ and measured u on ∂D (and we know v throughout D). We can solve equations (13) and (14) for the two quantities θ and $\int_\sigma [u]\, ds$, e.g., take the quotient of the left and right sides to find $\tan(\theta)$, from which one can determine θ (in $[0, \pi)$) and then $\int_\sigma [u]\, ds$.

The only potential difficulty is if $\int_\sigma [u]\, ds = 0$, in which case we cannot determine θ. However, the "odds" of this happening for randomly chosen boundary data are small (zero, really). Nonetheless, an issue worth thinking about is how to choose the input flux g so this integral is far from zero. We will henceforth assume that the input flux was chosen so that $\int_\sigma [u]\, ds \neq 0$.

Identifying the Crack Plane

We've now identified the angle at which the crack lies with respect to the x axis. To simplify further analysis, let's make a change of coordinates so that crack lies parallel to the x_1 axis (which we can do, now that we know the crack angle θ!), and hence in a plane of the form $x_2 = c$ for some constant c. Thus $\mathbf{n} = (0, 1)$ on σ.

To find c, let $v_3(x_1, x_2) = x_1^2 - x_2^2$; this function is harmonic. Note that $\frac{\partial v_3}{\partial \mathbf{n}} = -2c$ on σ. Putting v_3 into the reciprocity gap formula (12) yields

$$-2c \int_\sigma [u]\, ds = \int_{\partial D} (u \frac{\partial v_3}{\partial \mathbf{n}} - v_3 \frac{\partial u}{\partial \mathbf{n}})\, ds = RG(x_1^2 - x_2^2). \tag{15}$$

As long as $\int_\sigma [u]\, ds \neq 0$, we can solve for c (we found $\int_\sigma [u]\, ds$ above), and we've now identified the plane in which the crack lies.

Identifying the Crack Endpoints

First, let's again make a simple change of coordinates so that the crack lies on the x_1 axis (hence $\mathbf{n} = (0, 1)$ on σ); we can do this since we now know the line on which σ lies. Also, re-scale the coordinates so that the portion of the x_1 axis which passes through D is simply $0 < x_1 < 1$, as in the figure below:

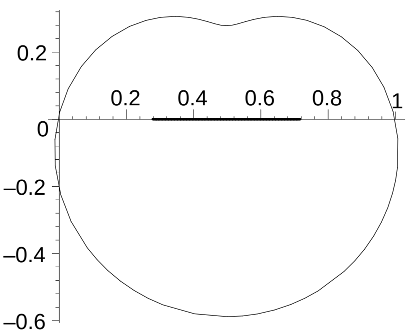

Figure 5

Note that the jump $[u]$ on σ can be though of as a function of x_1, but is defined only on the crack σ. However, we can think of extending $[u](x_1)$ to

$0 < x_1 < 1$ (the full width of D), by simply extending it as a zero function outside σ. So think of $[u](x_1)$ as a function defined on $D \cap \{x_2 = 0\}$, with $[u] \equiv 0$ outside σ. We are going to recover $[u]$ from the boundary data.

Define functions $v_k(x_1, x_2) = \frac{1}{k\pi}\sin(k\pi x_1)e^{k\pi x_2}$ for each integer $k \geq 1$. You can check that v_k is harmonic. Now note that $\frac{\partial v_k}{\partial \mathbf{n}} = \frac{\partial v_k}{\partial x_2} = \sin(k\pi x_1)$ on σ (where $x_2 \equiv 0$). Putting this function into equation (12) yields

$$\int_0^1 \sin(k\pi x_1)[u](x_1)\,dx_1 = \int_{\partial D}(u\frac{\partial v_k}{\partial \mathbf{n}} - v_k\frac{\partial u}{\partial \mathbf{n}})\,ds. \tag{16}$$

In setting up the integral on the left above, I've used the fact that we can integrate beyond the ends of σ, i.e., $\int_\sigma \sin(k\pi x_1)[u](x_1)\,ds = \int_0^1 \sin(k\pi x_1)[u](x_1)\,ds$, since $[u] \equiv 0$ beyond the end of the crack. Also, I've used that $ds = dx_1$.

Equation (15) shows that we can compute all the Fourier sine coefficients of the function $[u](x_1)$ defined for $0 < x_1 < 1$. This means we reconstruct the function $[u]$ itself, if you recall the fact from Fourier analysis that any L^2 function $\phi(x)$ defined on $0 < x < 1$ can be written as a sine series

$$\phi(x) = \sum_{k=1}^{\infty} c_k \sin(k\pi x)$$

where the c_k are given by

$$c_k = 2\int_0^1 \sin(k\pi x)\phi(x)\,dx$$

for $k \geq 1$. In our case the approximate jump reconstruction is given by

$$[u](x_1) = \sum_{k=1}^{n} c_k \sin(k\pi x_1) \tag{17}$$

with the c_k computed from equation (16), where we choose n as some appropriate finite number (taking n too big introduces numerically instability, as we can analyze later).

Thus the boundary data contains sufficient information (in principle, anyway) to recover $[u]$, but that's not quite (at first glance) the same as determining σ. Let S denote those points on the x_1 axis for which $[u] \neq 0$. We know that $[u] \equiv 0$ outside σ, and so certainly $S \subseteq \sigma$. But is it possible that S is strictly smaller than σ? This would require that $[u] \equiv 0$ on some open portion of σ. We show below that this is impossible.

In order to prove the claim we need two "big" results from PDE theory, that are often quite useful for inverse problems. The results are

Theorem 1: *Let w be a function which is harmonic in some two dimensional domain D and let γ be a simple smooth curve (once continuously differentiable) contained in D. Suppose that w and $\frac{\partial w}{\partial \mathbf{n}}$ are both identically zero on γ. Then w is identically zero throughout D.*

The result is also true if γ is a portion of ∂D.

Theorem 2: *Let w be a function which is harmonic in some two dimensional domain D. Suppose D is any open region (e.g., a ball) contained in D and $w \equiv 0$ on D. Then $w \equiv 0$ on all of D.*

Here's how to use the Theorems to finish the proof that $S = \sigma$: Suppose we could find some small open segment γ contained in σ such that $[u] \equiv 0$ on γ. We'll show that this leads to a contradiction. Define a function w in some neighborhood of the crack as $w(x_1, x_2) = u(x_1, x_2) - u(x_1, -x_2)$; note that ON σ we have $w = [u]$. You can check that

1. w is harmonic in some "one-sided" neighborhood R of γ.
2. $\frac{\partial w}{\partial \mathbf{n}} \equiv 0$ on γ.
3. $w \equiv 0$ on γ.

By Theorem 1 w must be identically zero in D, and hence by Theorem 2 w is identically zero in some neighborhood of the crack σ. But this means that $[u] \equiv 0$ in a neighborhood of σ, so that $\int_\sigma [u]\, ds = 0$ in contradiction to the assumptions made earlier. We conclude that $[u]$ cannot vanish identically on any open portion of σ, and so $S = \sigma$.

The completes the proof that a line segment crack σ can be recovered from the boundary data, and we in fact have a very constructive procedure to do this.

Computational Example

Let us take D as the unit disk in $\mathbb{R}^2$. The crack σ will be a line segment with one end at the point $(-0.3, 0.6)$, of length 0.2, lying at an angle of 0.4 radians with respect to horizontal. The situation is illustrated below:

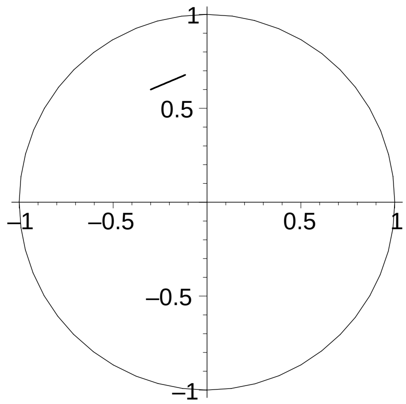

Figure 6

The input flux g applied in equation (6) is $g(\theta) = \sin(\theta)$, where the boundary of the disk is parameterized in the usual way, as $x = \cos(\theta), y = \sin(\theta)$ for $0 \leq \theta < 2\pi$. Heat energy enters on the top half of the domain and exits the bottom half. The solution u to equations (5)-(7) is computed via a standard numerical method; in particular, we compute or "measure" u on ∂D (at a total of 50 points). This allows us to compute $RG(v)$ as defined by equation (12) for any harmonic function v.

Equations (13) and (14) then become

$$-J\sin(\theta) = -0.01146609873, \quad J\cos(\theta) = .027119929$$

where $J = \int_\sigma [u]\, ds$. The unique solution to these equations with $-\pi/2 < \theta \leq \pi/2$ is

$$\theta = 0.4, \quad J = 0.029444. \tag{18}$$

The crack angle is actually accurate to 7 significant figures! The value of J is also good to 7 significant figures.

We can then rotate the entire region $D \setminus \sigma$ through an angle of -0.4 radians, to that σ is now known to be horizontal. To identify the y coordinate of the line on which σ lies we use equation (15), which becomes

$$-2cJ = - = .03942353783$$

which (given the already computed value of J) yields

$$c = 0.669 \tag{19}$$

accurate to at least 5 significant figures.

Finally, we identify the crack endpoints. Shift and rescale the domain $D \setminus \sigma$ as in Figure 5, and apply equation (16) for $k = 1$ to $k = 10$, the use equation (17) with $n = 10$. The resulting reconstruction looks like

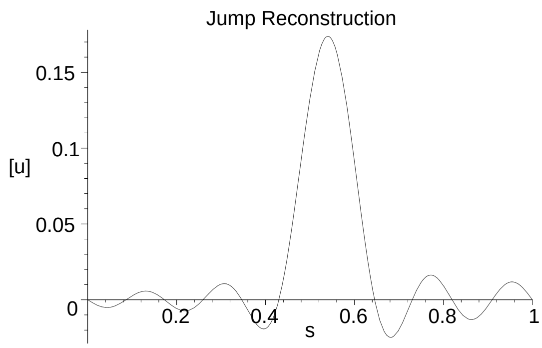

Figure 7

The crack should be the region where $[u] \neq 0$, but of course that can't be picked off with perfect clarity. We can "threshold" the jump function to approximate the crack location, in this case by estimating that the crack lies wherever $[u]$ exceeds 30 percent of the maximum value. This yields the estimate that the crack endpoints correspond to $s = 0.46$ and $s = 0.62$, roughly. We can know go back to the original unrotated/unscaled coordinates, to find that the estimated crack endpoints are $(-0.314, 0.594)$ (true was $(-0.3, 0.6)$) and $(-0.1, 0.648)$, corresponding to a length of 0.23. The estimated crack lies almost on top of the actual crack!