Volume in n Dimensions

MA 305

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Introduction

You've seen that if we have two vectors $\mathbf{v}$ and $\mathbf{w}$ in two dimensions then the area "spanned" by these vectors can be computed as

$$|\mathbf{v} \times \mathbf{w}| = |v_1 w_2 - v_2 w_1|$$

(where in the cross product we fake both vectors as 3D vectors by adding an artificial zero component for the z direction.) Given three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, the volume of the parallelpiped spanned can be computed as

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|,$$

which of course can be expanded out. The idea of the area spanned by two 2D vectors or the volume of three 3D vectors has been important in several applications. We've used it in setting up and computing the flux of a vector field over a curve (2D) or a surface (3D), as well as in computing surfaces areas, which were crucial steps in setting up the divergence and Stokes' Theorems. We also used these ideas in deriving the volume element dV in other coordinate systems.

How does this work in n dimensions? Specifically, given n vectors

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$$

in n dimensional space, what volume do they span? Note that I'm using subscripts for indexing the vectors—these are NOT components. The answer isn't too hard to figure out, especially if we start with two dimensions for inspiration.

Two Dimensions

Consider a function $\Delta_2(\mathbf{v}_1, \mathbf{v}_2)$ which takes as input 2 two-dimensional vectors and returns the "area" of the parallelogram spanned by the vectors. I put "area" in quotes, because the definition of Δ_2 that we cook up below will not be exactly the area, but very closely related. So pretend we don't know how to compute the area—we're going to figure it out with a bit of intuition, and this will serve as a guide for higher dimensions.

Intuitively, what properties should the "area" function Δ_2 have? First, it's obvious that

$$\Delta_2(\mathbf{v}, \mathbf{v}) = 0$$

for any vector $\mathbf{v}$, since no area is spanned. Second, the pictures below ought to convince you that if we fix one of the input vectors then the function Δ_2 is LINEAR in the remaining variable, that is,

$$\Delta_2(\mathbf{u}, \mathbf{v} + \mathbf{w}) = \Delta_2(\mathbf{u}, \mathbf{v}) + \Delta_2(\mathbf{u}, \mathbf{w}), \quad \text{and} \quad \Delta_2(\mathbf{u}, c\mathbf{v}) = c\Delta_2(\mathbf{u}, \mathbf{v}).$$

Or you can fix the second variable—then Δ_2 is linear in the first variable.

Finally, it's obvious that if $\mathbf{e}_1$ and $\mathbf{e}_2$ denote the unit vectors in the x and y directions, then
$$\Delta_2(\mathbf{e}_1, \mathbf{e}_2) = 1.$$
To summarize what we've said above, we want Δ_2 to have the properties

1. For any vector $\mathbf{v}$, $\Delta_2(\mathbf{v}, \mathbf{v}) = 0$.

2. If either variable is fixed, Δ_2 is linear in the remaining variable.

3. $\Delta_2(\mathbf{e}_1, \mathbf{e}_2) = 1$

Theorem 1: *There is one and only one function which satisfies properties 1 to 3 above.*

Before proving this it will be handy to prove

Proposition 1: *Any function which satisfies properties 1 to 3 above must satisfy*
$$\Delta_2(\mathbf{v}_1, \mathbf{v}_2) = -\Delta_2(\mathbf{v}_2, \mathbf{v}_1).$$

Note that this proposition shows that Δ_2 is not area, since Δ_2 can be negative.

To prove the proposition, first note that by property 1
$$\Delta_2(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2) = 0.$$
Now, using property 2 repeatedly we find that
$$\begin{aligned}
0 = \Delta_2(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2) &= \Delta_2(\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2) + \Delta_2(\mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2) \\
&= \Delta_2(\mathbf{v}_1, \mathbf{v}_1) + \Delta_2(\mathbf{v}_1, \mathbf{v}_2) + \Delta_2(\mathbf{v}_2, \mathbf{v}_1) + \Delta_2(\mathbf{v}_2, \mathbf{v}_2) \\
&= \Delta_2(\mathbf{v}_1, \mathbf{v}_2) + \Delta_2(\mathbf{v}_2, \mathbf{v}_1)
\end{aligned}$$

from which we immediately get Proposition 1.

Here is a proof of Theorem 1. It is constructive, in that we will not only show that there is a function consistent with properties 1 to 3, but also how to compute it. Before we begin it will be helpful to note that

$$\Delta_2(\mathbf{v}_1, c\mathbf{v}_1 + \mathbf{v}_2) = c\Delta_2(\mathbf{v}_1, \mathbf{v}_1) + \Delta_2(\mathbf{v}_1, \mathbf{v}_2) = \Delta_2(\mathbf{v}_1, \mathbf{v}_2). \tag{1}$$

In short, we can add a constant multiple of any one of the vectors to another input vector without changing the value of Δ_2.

In the case that $\mathbf{v}_1 = \mathbf{e}_1 = (1,0)$ and $\mathbf{v}_2 = \mathbf{e}_2 = (0,1)$ we know from property 3 how to compute $\Delta_2(\mathbf{v}_1, \mathbf{v}_2)$. We're going to reduce the general case to this special case by repeated application of rules 1 and 2. Let's first do a specific example so you see how it works, then the general case. Let $\mathbf{v}_1 = (3,2)$ and $\mathbf{v}_2 = (-2,7)$. Then repeated strategic application of equation (1) gives

$$
\begin{aligned}
\Delta_2(\mathbf{v}_1, \mathbf{v}_2) = \Delta_2((3,2),(-2,7)) &= \Delta_2((3,2), \frac{2}{3}(3,2) + (-2,7)) \\
&= \Delta_2((3,2),(0,25/3)) \\
&= \Delta_2((3,2) + \frac{-6}{25}(0,25/3),(0,25/3)) \\
&= \Delta_2((3,0),(0,25/3)) \\
&= 25\Delta_2((1,0),(0,1)) \\
&= 25.
\end{aligned}
$$

Notice that at each stage I choose to multiply a given vector by just the right constant so that when the result is added to the other vector something is cancelled.

If we apply the above strategy to the general case in which $\mathbf{v}_1 = (a_1, a_2)$ and $\mathbf{v}_2 = (b_1, b_2)$ we obtain

$$
\begin{aligned}
\Delta_2(\mathbf{v}_1, \mathbf{v}_2) &= \Delta_2((a_1, a_2),(b_1, b_2)) \\
&= \Delta_2((a_1, a_2), \frac{-b_1}{a_1}(a_1, a_2) + (b_1, b_2)) \\
&= \Delta_2((a_1, a_2),(0, \frac{a_1 b_2 - b_1 a_2}{a_1})) \\
&= \Delta_2((a_1, a_2) + \frac{-a_1 a_2}{a_1 b_2 - a_2 b_1}(0, \frac{a_1 b_2 - b_1 a_2}{a_1}),(0, \frac{a_1 b_2 - b_1 a_2}{a_1})) \\
&= \Delta_2((a_1, 0),(0, \frac{a_1 b_2 - b_1 a_2}{a_1})) \\
&= (a_1 b_2 - b_1 a_2)\Delta_2((1,0),(0,1)) \\
&= a_1 b_2 - b_1 a_2.
\end{aligned}
$$

This proves that $\Delta_2((a_1, a_2),(b_1, b_2)) = a_1 b_2 - a_2 b_1$ is the only formula which can be consistent with properties 1 to 3. In fact, you can check explicitly that it IS consistent with them.

Now, Δ_2 is not itself the area, for the simple reason that area is nonnegative. But in fact, you should recognize that the area spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ is just $|\Delta_2(\mathbf{v}_1, \mathbf{v}_2)|$.

If you've studied any linear or matrix algebra then you recognize the formula for Δ_2 as being that for the determinant of the 2 by 2 matrix

$$
\begin{bmatrix}
a_1 & a_2 \\
b_1 & b_2
\end{bmatrix}
$$

The Case in n Dimensions

With two dimensions as inspiration, let's generalize to n dimensions (think of 3D to see what's going on.) We will define a function

$$\Delta_n(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$$

which accepts n vectors (all n-dimensional) and returns what we might call the "signed" n dimensional volume spanned by the vectors (meaning that it is the volume to within a plus or minus sign.) If you think about it (certainly in 3D) we should demand exactly the same properties from Δ_n that we demanded from Δ_2, namely

1. If any two of the input vectors are identical (so they're parallel) then $\Delta_n = 0$, i.e.,

$$\Delta_n(\mathbf{v}_1, \ldots, \mathbf{w}, \ldots, \mathbf{w}, \ldots, \mathbf{v}_n) = 0.$$

2. If $n - 1$ of the input vectors are fixed then Δ_n is linear in the remaining variable.

3. $\Delta_n(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n) = 1$.

Just as for two dimensions we can prove

Theorem 2: *There is one and only one function Δ_n which satisfies properties 1 to 3 above.*

We will actually show that there is AT MOST one such function, and show how it would have to be computed. The proof that the resulting formula or procedure really is well-defined and consistent with properties 1 to 3 is messy and not very enlightening and so, to quote many authors who don't like to write such stuff out, it is "left to the reader."

One thing worth noting is that Proposition 1 is still valid. Specifically, we have

Proposition 1': *For any function Δ_n which satisfies properties 1 to 3 above, interchanging any two vectors in Δ_n reverses the sign of Δ_n.*

The proof is almost exactly like the 2D case (first try the case in which $\mathbf{v}_1$ and $\mathbf{v}_2$ are interchanged, and you'll see that any other case is basically the same).

Here is a computation which shows that there is at most one function which meets the three criteria above, and shows how such a function must be computed. In fact, rather than write out the general case, let's do a specific example in three dimensions. You will see how to do it in higher dimensions, and also why I didn't want to write out a bigger case. Consider the three vectors

$$\mathbf{v}_1 = (1, 4, 2), \quad \mathbf{v}_2 = (1, 4, -4), \quad \mathbf{v}_3 = (3, -2, -4).$$

First, it will be very convenient to write the vectors as the rows of a 3 by 3 matrix, i.e., as

$$\begin{bmatrix} 1 & 4 & 2 \\ 1 & 4 & -4 \\ 3 & -2 & -4 \end{bmatrix}$$

Using this notation we can think of Δ_n as a function which acts on n by n matrices. I will put absolute value signs around the matrix as short hand for Δ_n applied to the corresponding vectors.

Now, I want to multiply the first vector in $\Delta_3(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ by -3 and add the result to the last vector, because this will not change the value of Δ_n:

$$\Delta_3(\mathbf{v}_1, \mathbf{v}_2, -3\mathbf{v}_1 + \mathbf{v}_3) = -3\Delta_3(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_1) + \Delta_3(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$$

(by properties 1 and 2). Also, it will "zero out" the first component of the last vector. In matrix terms, I want to multiply the first row by -3 and add it to the last row. This shows that

$$\left| \begin{bmatrix} 1 & 4 & 2 \\ 1 & 4 & -4 \\ 3 & -2 & -4 \end{bmatrix} \right| = \left| \begin{bmatrix} 1 & 4 & 2 \\ 1 & 4 & -4 \\ 0 & -14 & -10 \end{bmatrix} \right|$$

Our goal is to use this kind of "multiply a row by a constant and add to another row" procedure to eventually produce a matrix with zeros everywhere except on the diagonal. With this in mind I'll now multiply the first row by -1 and add the result to the second row. By the same reasoning as above, this will not change the value of Δ_3 and we'll have

$$\left| \begin{bmatrix} 1 & 4 & 2 \\ 1 & 4 & -4 \\ 3 & -2 & -4 \end{bmatrix} \right| = \left| \begin{bmatrix} 1 & 4 & 2 \\ 0 & 0 & -6 \\ 0 & -14 & -10 \end{bmatrix} \right|$$

Notice now that I've eliminated all of the entries in the first column, except the one on the diagonal at the upper left. I'd like now to employ the same procedure to eliminate all of the entries in the second column, except the one on the "diagonal" (second row, second column). However, as it stands this is impossible—multiplying the second row by any scalar and adding to rows 1 or 3 will not get rid of the second column entries.

The key here is to *pivot*, that is, interchange two rows to get something nonzero into the second column/second row position. From Proposition 1' we know that this will reverse the sign of Δ_n. Interchanging row 2 and row 1 will spoil the zero element we obtained in the lower left corner, so we'll interchange rows 2 and 3 to find that

$$\left| \begin{bmatrix} 1 & 4 & 2 \\ 1 & 4 & -4 \\ 3 & -2 & -4 \end{bmatrix} \right| = -\left| \begin{bmatrix} 1 & 4 & 2 \\ 0 & -14 & -10 \\ 0 & 0 & -6 \end{bmatrix} \right| \tag{2}$$

Now proceed as before: Multiply row 2 by $\frac{4}{14}$ and add the result to row 1 to get

$$\left| \begin{bmatrix} 1 & 4 & 2 \\ 1 & 4 & -4 \\ 3 & -2 & -4 \end{bmatrix} \right| = -\left| \begin{bmatrix} 1 & 0 & -\frac{6}{7} \\ 0 & -14 & -10 \\ 0 & 0 & -6 \end{bmatrix} \right|$$

Finally, multiply the last row by appropriate scalars and add to rows 1 and 2 to obtain

$$\left| \begin{bmatrix} 1 & 4 & 2 \\ 1 & 4 & -4 \\ 3 & -2 & -4 \end{bmatrix} \right| = -\left| \begin{bmatrix} 1 & 0 & 0 \\ 0 & -14 & 0 \\ 0 & 0 & -6 \end{bmatrix} \right|$$

In other words, we've shown that $\Delta_3(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is $-\Delta_3(\mathbf{e}_1, -14\mathbf{e}_2, -6\mathbf{e}_3)$, and the latter is by property 2 above (linearity) equal to $-84\Delta_3(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, which by inspection equals -84. Thus $\Delta_3(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = -84$ and the volume spanned by the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is equal to $|-84| = 84$.

This procedure can be carried out methodically for any n vectors in n dimensions, and this constructive procedure shows that there is at MOST one function consistent with properties 1 to 3. It doesn't quite show that such a function exists. For although we have a procedure to compute Δ_n, it's not clear that the function is well-defined; if we had done the procedure in a different order (there were many possible choices for how to eliminate entries in the matrix) would we have obtained the same result? Even if that were the case, is the resulting function Δ_n actually consistent with properties 1 to 3? Fortunately, it turns out that Δ_n is well-defined and consistent with the specified properties. We thus have a method for finding the volume spanned by n vectors in n dimensional space.

You might recognize Δ_n as the determinant of the n by n matrix formed from the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, and the procedure we used for computing the determinant as simply Gaussian elimination. In fact, our procedure is somewhat overkill—we didn't need to zero out the matrix entries above the diagonal. We could have stopped with equation (2) and found Δ_n by simply reading off the product of the entries along the diagonal of the matrix on the right of equation (2). You should think about why this is true.

By the way, Gaussian elimination is not a bad way to compute the determinant. It is infinitely more efficient than "expansion by minors", a technique you might well have seen before. For an n by n matrix Gaussian elimination is $O(n^3)$ efficient, while expansion by minors is $O(n!)$.

Change of Variables and Integrals in Other Coordinate Systems

Let $(x_1, \ldots, x_n)$ be rectangular coordinates for n dimensional space. Suppose we have another coordinate system with coordinates $(u_1, \ldots, u_n)$. Just as for polar or cylindrical or spherical, the coordinates in each system can be thought of as functions of the coordinates in the other system, e.g.,

$$x_1 = x_1(u_1, \ldots, u_n) \tag{3}$$
$$\vdots \tag{4}$$
$$x_n = x_n(u_1, \ldots, u_n) \tag{5}$$

Suppose that you change the u_i variable by a small amount du_i while holding the other u_k constant. In doing so the x_j variable would change by a small amount

$$dx_j = \frac{\partial x_j}{\partial u_i} du_i$$

for $j = 1$ to n. Let $d\mathbf{x}_i$ denote the position change due to changing u_i by an amount du_i. Then

$$\begin{aligned} d\mathbf{x}_i &= dx_1\mathbf{e}_1 + dx_2\mathbf{e}_2 + \cdots + dx_n\mathbf{e}_n \\ &= \left(\frac{\partial x_1}{\partial u_i}\mathbf{e}_1 + \cdots \frac{\partial x_n}{\partial u_i}\mathbf{e}_n \right) du_i \end{aligned} \tag{6}$$

where $\mathbf{e}_k$ is the unit vector in the x_k direction.

Now consider the result of changing each of the u_i by a small amount du_i from some "base" point. The resulting n vectors $d\mathbf{x}_1, \ldots, d\mathbf{x}_n$ would span some kind of n dimensional parallelogram, and from our previous computation its volume would be

$$dV = |\Delta_n(d\mathbf{x}_1, \ldots, d\mathbf{x}_n)|\, du_1 \cdots, du_n = \left| \det \begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \cdots & \frac{\partial x_n}{\partial u_1} \\ \vdots & & \\ \frac{\partial x_1}{\partial u_n} & \cdots & \frac{\partial x_n}{\partial u_n} \end{bmatrix} \right| du_1 \cdots, du_n$$

where "det" is the determinant. The matrix above is the transpose of what is called the *Jacobian matrix*. (It would actually be the Jacobian if we had done our vectors as columns.) Thus if you want to integrate a function f over a region D and use the coordinate system $u_1, \ldots, u_n$ you should compute

$$\int_D f\, dV$$

where dV is given as above. This will yield the same value for the integral in any coordinate system.

Example: As a specific example, consider the relations between rectangular and spherical coordinates:

$$\begin{aligned} x &= \rho \cos(\theta) \sin(\phi) \\ y &= \rho \sin(\theta) \sin(\phi) \\ z &= \rho \cos(\phi) \end{aligned}$$

In this case you find that (taking the spherical variables in the order ρ, θ, ϕ)

$$dV = \left| \det \begin{bmatrix} \cos(\theta)\sin(\phi) & \sin(\theta)\sin(\phi) & \cos(\phi) \\ -\rho\sin(\theta)\sin(\phi) & \rho\cos(\theta)\sin(\phi) & 0 \\ \rho\cos(\theta)\cos(\phi) & \rho\sin(\theta)\cos(\phi) & -\rho\sin(\phi) \end{bmatrix} \right| d\rho\, d\theta\, d\phi,$$

which works out to $dV = \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi$, thus usual formula.

Exercise

- Compute the volume spanned by three vectors $\mathbf{v}_1 = (a_1, a_2, a_3)$, $\mathbf{v}_2 = (b_1, b_2, b_3)$, and $\mathbf{v}_3 = (c_1, c_2, c_3)$ and verify that it is consistent with the three properties above.