Solvability of Laplace's Equation

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1 Introduction

Let D be a bounded region in $\mathbb{R}^n$, with $x = (x_1, \ldots, x_n)$. We seek a function $u(x)$ which satisfies

$$\begin{aligned} \triangle u &= 0 \text{ in } D, &(1)\\ u &= h \text{ on } \partial D &(2)\\ &\text{OR}\\ \frac{\partial u}{\partial n} &= g \text{ on } \partial D. &(3) \end{aligned}$$

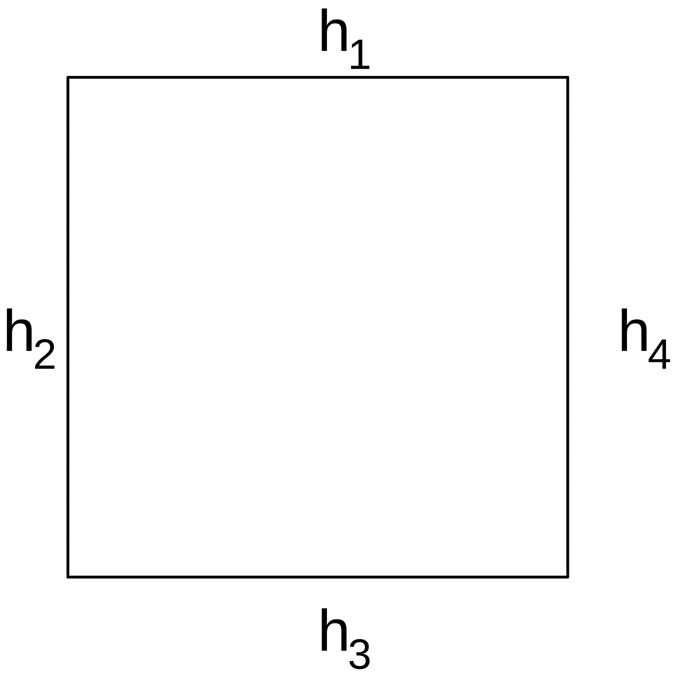
We've shown that if a solution exists, it's unique (though only up to an additive constant for the Neumann boundary condition; in this case there's also a requirement on g). We've also shown other interesting properties possessed by any solution, e.g., the maximum principle and the mean value property.

Existence is actually the toughest issue. On certain special domains like rectangles and circles/spheres it becomes much easier. We'll look at those first, then at the more general problem.

2 Solving Laplace's Equation

2.1 Rectangles

Let D be the rectangle (square, really) $0 \leq x, y \leq 1$ in two dimensions, where I'm now using conventional (x, y) coordinates, instead of (x_1, x_2).



You'll easily see how to generalize what we do to other rectangular two-dimensional regions, and even n dimensions. Let's look at the case of Dirichlet boundary conditions, so we want a function $u(x,y)$ defined for $0 \le x, y \le 1$ with $u = h$ for some function h defined on ∂D. Let's split h into four parts: h_1 will be defined as equal to h on the top of the rectangle and be zero on the other three sides; similarly for h_2, h_3, and h_4, as illustrated above. We'll solve Laplace's equation by splitting the solution u into four corresponding parts, $u = u_1 + u_2 + u_3 + u_4$ where $\triangle u_j = 0$ and $u_j = h_j$ on ∂D.

Let's start by solving for u_1. The other three pieces will obviously be similar. Separate variables by assuming that a solution to Laplace's equation can be written as $u_1(x,y) = X(x)Y(y)$. Plug this into Laplace's equation to find that

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} = \lambda$$

for some constant λ. The constant λ can be positive, negative, or zero. Let's assume for the moment that λ is positive; you can verify that when we're looking for u_1 or u_3 this is the only interesting choice, while if we're looking for u_2 or u_4 then we'd take $\lambda < 0$. For positive λ the general solution for X

and Y is

$$\begin{aligned} X(x) &= c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x), \\ Y(y) &= c_3 e^{\sqrt{\lambda}y} + c_4 e^{-\sqrt{\lambda}y} \end{aligned}$$

for some constants c_1, c_2, c_3, c_4. We need $u_1(0, y) = 0$ (this is the left side). This instantly forces $c_1 = 0$. Also, then condition that $u_1(1, y) = 0$ on the right forces $\lambda = k^2\pi^2$. We conclude that $X(x) = c_2 \sin(k\pi x)$ for some integer k. Finally, the condition that $u_1(x, 0) = 0$ on the bottom forces us to choose $c_3 = -c_4$. All in all then the product $X(x)Y(y)$ looks like

$$X(x)Y(y) = c\sin(k\pi x)(e^{k\pi y} - e^{-k\pi y})$$

for some integer k, where I've lumped all constants together into one constant, c. Of course, any linear combination of solutions will again be a solution. We ought to take $u_1(x, y)$ of the form

$$u_1(x, y) = \sum_{k=1}^{\infty} c_k \sin(k\pi x)(e^{k\pi y} - e^{-k\pi y}). \tag{4}$$

Exercise: Suppose we're looking for the harmonic function on D with Dirichlet data $h_2(y)$ on the right ($x = 1$) side of D; why should we take $\lambda < 0$ in $-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} = \lambda$? What equation is obtained in place of (4)?

How do we get the constants c_k in equation (4)? With the boundary condition, $u_1(x, 1) = h_1(x)$, of course! Note that $h_1(x)$ can be expand into a sine series in x, as

$$h_1(x) = \sum_{k=1}^{\infty} d_k \sin(k\pi x) \tag{5}$$

where

$$d_k = 2\int_0^1 h_1(x)\sin(k\pi x)\,dx$$

for $k \geq 1$. From equation (4) we also find that

$$u_1(x, 1) = \sum_{k=1}^{\infty} c_k \sin(k\pi x)(e^{k\pi} - e^{-k\pi}). \tag{6}$$

Match coefficients on the right of (5) and (6) to find that if we take $c_k = \frac{d_k}{e^{k\pi}-e^{-k\pi}}$ in equation (4) we have the solution $u_1(x, y)$ given by

$$u_1(x, y) = \sum_{k=1}^{\infty} d_k \frac{e^{k\pi y} - e^{-k\pi y}}{e^{k\pi} - e^{-k\pi}} \sin(k\pi x). \tag{7}$$

But if you've been awake in the course so far you should realize that there is the slightly delicate issue of whether this series solution (7) really makes any sense, i.e., converges and defined a truly differentiable function with the correct boundary values. Recall the following Theorem, which we used in analyzing the heat equation:

Theorem 1 *Let ϕ_k, $k \geq 1$, be a sequence of functions defined on an interval $[a, b]$. Suppose that each $\phi_k \in C^1([a, b])$. Let $M_k = \sup_{a<x<b} |\phi_k'|$. If $\sum_{k=1}^{\infty} M_k < \infty$ then the series $\sum_k \phi_k'$ and $\sum_k \phi_k$ converge uniformly on $[a, b]$. If $\phi = \sum_k \phi_k$ then $\phi \in C^1([a, b])$ and*

$$\phi'(x) = \sum_k \phi_k'(x)$$

for $a \leq x \leq b$.

The theorem can be used to show that the series solution (7) is differentiable in both x and y, in fact, infinitely differentiable. To see this, think of y as fixed, $0 < y < 1$, and the series (7) as a series in $\sin(k\pi x)$. Apply the theorem above with $\phi_k(x) = k\pi d_k \frac{e^{k\pi y}-e^{-k\pi y}}{e^{k\pi}-e^{-k\pi}} \sin(k\pi x)$; note that we would then take

$$M_k = \left| k\pi d_k \frac{e^{k\pi y} - e^{-k\pi y}}{e^{k\pi} - e^{-k\pi}} \right|.$$

It's not hard to show that (since $\sum_k d_k^2 < \infty$, so the d_k are bounded) the series $\sum_k M_k$ converges (because the terms $\frac{e^{k\pi y}-e^{-k\pi y}}{e^{k\pi}-e^{-k\pi}}$ die off very rapidly with respect to k if $y \in (0, 1)$.) Thus $u_1(x, y)$ is differentiable term-by-term in x, and you can apply the argument again to get second derivatives (or higher) in x. A very similar argument (fix x, treat u_1 as a series in y) works to show that u_1 is differentiable in y, so (7) really does defined a function which is C^2.

Exercise: Fill in the details that $\sum_k M_k < \infty$.

As for the boundary values, an elementary argument quite similar to what we did for the 1D heat equation (and which we'll repeat on circular domains momentarily anyway) shows that $\lim_{t\to 0^+} u(x,t) = f(x)$ for our tentative solution $u(x,t)$) shows that

$$\lim_{x\to 0^+} \|u_1(x,y)\|_2 = \lim_{x\to 1^-} \|u_1(x,y)\|_2 = \lim_{y\to 0^+} \|u_1(x,y)\|_2 = 0$$

where $\|\phi\|_2$ is the $L^2(0,1)$ norm of ϕ. Also,

$$\lim_{y\to 1^-} \|u_1(x,y) - h_1(x)\|_2 = 0.$$

So u_1 has the correct boundary values, in the sense of L^2 distance. If h_1 is smooth enough you can replace the L^2 norms with the supremum norm.

As remarked above, the same procedure could clearly be done for the other three sides, and for any other rectangular region. This shows the existence of a solution to Laplace's equation on a rectangular region, provided that the Fourier series converge rapidly enough.

2.2 Circular Regions and the Laplacian in Polar Coordinates

Dealing with Laplace's equation on circles is much easier if we switch to polar coordinates, so that all derivatives will be with respect to r and θ, not x and y. You should recall that the formulae relating rectangular and polar coordinates are $x = r\cos(\theta)$, $y = r\sin(\theta)$ (polar to rectangular) and $r = \sqrt{x^2+y^2}$, $\theta = \pm\arctan(y,x)$ (rectangular to polar). The function $\arctan(y,x)$ is defined as $\arctan(y/x)$ for (x,y) in the first or fourth quadrants, and as $\arctan(y/x)+\pi$ or $\arctan(y/x)-\pi$ for (x,y) in the second and third quadrants, respectively.

Suppose that $u(x,y)$ is any function defined in terms of rectangular coordinates. We consider a "new" function v, identical to u, but defined in polar coordinates according to $v(r,\theta) = u(x,y)$. If we use the relations between polar and rectangular coordinates this is the same as

$$u(x,y) = v(\sqrt{x^2+y^2}, \arctan(y,x)).$$

Differentiate both sides above with respect to x. With the chain rule you find that

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{x}{\sqrt{x^2+y^2}}\frac{\partial v}{\partial r} - \frac{y}{x^2+y^2}\frac{\partial v}{\partial \theta}, \\ &= \cos(\theta)\frac{\partial v}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial v}{\partial \theta}. \end{aligned}$$

Put into differential operator form, this tells us how to translate $\frac{\partial}{\partial x}$ into polar coordinates, as

$$\frac{\partial}{\partial x} = \cos(\theta)\frac{\partial}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta}. \tag{8}$$

For example, consider the function $u(x, y) = x^2$. Then $\frac{\partial u}{\partial x} = 2x$. In polar coordinates we have $u(r, \theta) = r^2 \cos^2(\theta)$. Apply the differential operator on the right side of equation (8) to obtain $2r\cos^3(\theta) + 2r\sin^2(\theta)\cos(\theta) = 2r\cos(\theta)(\cos^2(\theta) + \sin^2(\theta)) = 2r\cos(\theta)$ which is just $2x$ back in rectangular coordinates.

The same procedure applied to $\frac{\partial}{\partial y}$ shows that

$$\frac{\partial}{\partial y} = \sin(\theta)\frac{\partial}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}. \tag{9}$$

We compute $\frac{\partial^2}{\partial x^2}$ in polar coordinates by applying $\frac{\partial}{\partial x}$ to itself (all in polar, of course). If the abstract differential operator form bothers you, stick in some unspecified function $v(r, \theta)$ when you do the derivatives. In any case, you'll get a BIG mess for both $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^2}{\partial y^2}$. However, when you add and start cancelling you find that

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{10}$$

This is the Laplacian in polar coordinates.

2.3 Solving Laplace's Equation on a Disk

Given the form of the Laplacian in polar coordinates, it's natural to look for harmonic functions which are separable in polar coordinates, that is, $v(r, \theta) =$

$f(r)g(\theta)$. If you plug this into Laplace's equation in polar coordinates you obtain

$$f''(r)g(\theta) + \frac{1}{r}f'(r)g(\theta) + \frac{1}{r^2}f(r)g''(\theta) = 0.$$

Multiply through by $\frac{r^2}{f(r)g(\theta)}$ and this becomes

$$r^2\frac{f''(r)}{f(r)} + r\frac{f'(r)}{f(r)} + \frac{g''(\theta)}{g(\theta)} = 0.$$

The usual argument shows that we must have

$$r^2\frac{f''(r)}{f(r)} + r\frac{f'(r)}{f(r)} = \lambda, \tag{11}$$

$$\frac{g''(\theta)}{g(\theta)} = -\lambda \tag{12}$$

for some constant λ. The choice $\lambda < 0$ turns out not to be useful (try it). If $\lambda > 0$ then we obtain

$$g(\theta) = c_1 \cos(\sqrt{\lambda}\theta) + c_2 \sin(\sqrt{\lambda}\theta).$$

But if we want the solution to be periodic (so that $v(r,\theta)$ is continuous) then we need $g(0) = g(2\pi)$. This forces us to choose $\sqrt{\lambda} = k$, or $\lambda = k^2$, where k is any integer (even 0 or negative), so we have

$$g(\theta) = c_1 \cos(k\theta) + c_2 \sin(k\theta).$$

The general solution to equation (11) is $f(r) = c_3 r^{\sqrt{\lambda}} + c_4 r^{-\sqrt{\lambda}}$. With $\lambda = k^2$ this becomes

$$f(r) = c_3 r^k + c_4 r^{-k}.$$

The plus and minus values of k are redundant—since k can be an arbitrary integer, we might as well just take $f(r) = cr^k$. All in all, any functions of the form

$$\phi(r,\theta) = r^k \cos(k\theta), \quad \phi(r,\theta) = r^k \sin(k\theta) \tag{13}$$

are harmonic, for any integer k. Of course if $k < 0$ then these solutions are singular at the origin.

As remarked above $\lambda < 0$ in (11) and (12) leads nowhere. However, we can also take $\lambda = 0$. In this case we find that if g is to be periodic then we must take $g(\theta)$ to be constant. Equation (11) also becomes easy to solve in this case, and the solution is $f(r) = c_1 + c_2 \ln(r)$. Take the product $f(r)g(\theta)$ to see that anything of the form

$$u(r, \theta) = c_1 + c_2 \ln(r) \tag{14}$$

is also harmonic. If $c_2 \neq 0$ then this solution is singular at the origin—in fact, with $c_1 = 0$ and $c_2 = \frac{1}{2\pi}$ we obtain the Green's function.

We can use the harmonic functions in (13) and (14) to solve Laplace's equation on a disk in two dimensions. For simplicity, let's use the unit disk. Let's parameterize the boundary of the disk as $(\cos(\theta), \sin(\theta))$ for $0 \leq \theta < 2\pi$. We want to find $v(r, \theta)$ so that $\triangle v = 0$ and $v(1, \theta) = h(\theta)$ for some specified boundary data h, which we'll assume is in $L^2(0, 2\pi)$. We'll try to write the solution as a sum of the basic solutions given in (13) and (14), as

$$v(r, \theta) = b_0 + \sum_{k=1}^{\infty} a_k r^k \sin(k\theta) + b_k r^k \cos(k\theta). \tag{15}$$

(Why did we exclude the $\ln(r)$ and $k < 0$ cases?) If we want $v(1, \theta) = h(\theta)$ then what we really need is

$$v(1, \theta) = b_0 + \sum_{k=1}^{\infty} a_k \sin(k\theta) + b_k \cos(k\theta) = h(\theta).$$

In other words, we need to expand h in terms of sines and cosines! We've already done all this. The formulas are

$$\begin{aligned} b_0 &= \frac{1}{2\pi} \int_0^{2\pi} h(\theta)\, d\theta, && (16) \\ a_k &= \frac{1}{\pi} \int_0^{2\pi} \sin(k\theta) h(\theta)\, d\theta, && (17) \\ b_k &= \frac{1}{\pi} \int_0^{2\pi} \cos(k\theta) h(\theta)\, d\theta. && (18) \end{aligned}$$

With this choice for the coefficients, equation (15) represents a solution to Laplace's equation on the disk, provided that the infinite series makes sense.

The solution (15) really does represent a C^2 solution to the problem. To see this we use Theorem 1. First, let's just consider the solution (15) with the cosine terms (we can do what follows for the sine separately, then put the results together). First we'll show that v is differentiable in θ. Fix a value of r with $0 \leq r < 1$ and let $\phi_k(\theta) = b_k r^k \cos(k\pi\theta)$, so $\phi_k'(\theta) = -k\pi b_k r^k \sin(k\pi\theta)$. We can take $M_k = k\pi|b_k|r^k$. Now if the function h is L^2 then $\sum_k b_k^2 < \infty$ and then $\sup_k |b_k| \leq B$ for some B. In this case

$$\sum_{k=1}^{\infty} M_k \leq B\pi \sum_{k=1}^{\infty} kr^k = \frac{B\pi r}{(r-1)^2}.$$

Thus $v(r,\theta)$ is differentiable in θ. Repeat the argument on the series expansion of $\frac{\partial v}{\partial \theta}$ to find v is twice (in fact, infinitely) differentiable in θ. A similar argument works to show v is infinitely differentiable in r for $0 \leq r < 1$.

Let's also take a look at the behavior of v at the boundary. I claim that $\lim_{r\to 1^-} \|v(r,\theta) - h(\theta)\|_2 = 0$, so v has the correct Dirichlet data in the L^2 sense. To see this note that we have

$$h(\theta) - v(r,\theta) = \sum_{k=1}^{\infty}(a_k(1-r^k)\sin(k\theta) + b_k(1-r^k)\cos(k\theta)). \tag{19}$$

The orthogonality and completeness of the $\sin(k\theta), \cos(k\theta)$ family shows that

$$\|h(\theta) - v(r,\theta)\|_2^2 = \sum_{k=1}^{\infty} c_k^2(1-r^k)^2 \tag{20}$$

where $c_k^2 = a_k^2 + b_k^2$. Now since $h \in L^2(0, 2\pi)$ we have $\sum_k c_k^2 = \|h\|_2^2 < \infty$. As a result, given $\epsilon > 0$ we can choose some integer N such that

$$\sum_{k=N+1}^{\infty} c_k^2 < \epsilon/2.$$

It follows that (since $0 < (1-r^k)^2 < 1$)

$$\sum_{k=N+1}^{\infty} (1-r^k)^2 c_k^2 < \epsilon/2 \tag{21}$$

for all r. From equation (20) we can conclude that

$$\|h(\theta) - v(r,\theta)\|_2^2 = \sum_{k=1}^{N} c_k^2(1-r^k)^2 + R(\epsilon) \tag{22}$$

where $R(r) = \sum_{k=N+1}^{\infty}(1-r^k)^2 c_k^2 < \epsilon/2$. We can also choose some δ so that

$$\sum_{k=1}^{N} c_k^2(1-r^k)^2 < \epsilon/2 \tag{23}$$

for whenever $|1-r| < \delta$. All in all we conclude from equations (21)-(23) that for any given $\epsilon > 0$ we can choose a δ such that for $|1-r| < \delta$ we have

$$\|h(\theta) - v(r,\theta)\|_2^2 < \epsilon.$$

This is the definition of $\lim_{r\to 1^-} \|h(\theta) - v(r,\theta)\|_2^2 = 0$.

Exercise: Derive a similar formula to (15) for constructing a harmonic function $v(r,\theta)$ on the unit disk with boundary condition $\frac{\partial v}{\partial n} = h(\theta)$ for some function h (note that on the boundary of the disk $\frac{\partial v}{\partial n} = \frac{\partial v}{\partial r}$). Where does the requirement $\int_{\partial D} h\, ds = 0$ come into play?

2.4 Boundary Regularity

In PDE the term "regularity" has nothing to do with the bathroom. Rather, it refers to how smooth or differentiable the solution to a PDE is. We've shown that the solution v to Laplace's equation we found above is twice-differentiable inside the disk, and so we really can plug v into $\triangle v = 0$. But the statement that $v = h$ on ∂D is true only in the sense that $\lim_{r\to 1^-} \|h(\theta) - v(r,\theta)\|_2 = 0$, not quite as strong as one might hope for. It would be more "natural" if we could assert that $\lim_{r\to 1^-} \|h(\theta) - v(r,\theta)\|_\infty = 0$, so that in particular $u(r,\theta)$ converges to $h(\theta)$ as $r \to 1$. This isn't true, though, without additional conditions on the function h. To see this simply write out a solution to Laplace's equation with boundary data $h(\theta) = 0$ for $0 \le \theta < \pi$, $h(\theta) = 1$ for $\pi \le \theta < 2\pi$. The function h is L^2, but the solution defined by (15) won't converge to h at the discontinuity at $\theta = \pi$.

But if the function h is in $C^2([0, 2\pi])$, that is, h has two continuous derivatives everywhere on the boundary of the circle (including across $\theta = 2\pi$) then we can assert that $\lim_{r\to 1^-} \|h(\theta) - v(r,\theta)\|_\infty = 0$. (Actually, you can get away with h being C^1, but this is harder to prove). The following Lemma will be useful for this purpose. It's true on any interval $[a, b]$, but I'll just state it for $[0, 2\pi]$.

Lemma 1 *Suppose a function $h \in C^2([0, 2\pi])$ with $h(0) = h(2\pi)$ and $h'(0) = h'(2\pi)$. Let h_n be the n term Fourier series for h, i.e.,*

$$h_n(x) = b_0 + \sum_{k=1}^{n} (a_k \sin(kx) + b_k \cos(kx)) \tag{24}$$

where the a_k and b_k are defined by equations (18). Then the functions h_n converge uniformly to h on $[0, 2\pi]$, i.e.,

$$\lim_{n\to\infty} \|h - h_n\|_\infty = 0.$$

Proof: The functions h' and h'' are continuous, hence in $L^2(a, b)$, and so have Fourier expansions

$$\begin{aligned} h'(x) &= b_0' + \sum_{k=1}^{n} (a_k' \sin(kx) + b_k' \cos(kx)) & (25) \\ h''(x) &= b_0'' + \sum_{k=1}^{n} (a_k'' \sin(kx) + b_k'' \cos(kx)). & (26) \end{aligned}$$

Here the notation a_k' simply denotes the corresponding coefficient of h'—the prime isn't any kind of derivative. It's easy to work out these coefficients and relate them to the a_k and b_k via integration by parts, e.g.,

$$\begin{aligned} a_k'' &= \frac{1}{\pi} \int_0^{2\pi} h''(x) \sin(kx)\, dx \\ &= -\frac{k}{\pi} \int_0^{2\pi} h'(x) \cos(kx)\, dx = -k b_k' \\ &= -\frac{k^2}{\pi} \int_0^{2\pi} h(x) \sin(kx)\, dx \\ &= -k^2 a_k. \end{aligned}$$

In the integration by parts all endpoints terms are zero due to the periodicity of the trig functions AND h or h'. In particular we have $b_k' = -a_k''/k$ and $a_k' = b_k''/k$ (and it turns out that $b_0' = b_0'' = 0$).

Now I claim that $\sum_k (|a_k'| + |b_k'|) < \infty$. To see this note that

$$\sum_k (|a_k'| + |b_k'|) = \sum_k \frac{|a_k''|}{k} + \sum_k \frac{|b_k''|}{k}$$

$$
\begin{aligned}
&\leq \left(\sum_k 1/k^2\right)^{1/2}\left[\left(\sum_k (a_k'')^2\right)^{1/2} + \left(\sum_k (b_k'')^2\right)^{1/2}\right] \\
&\leq \frac{\pi}{\sqrt{6}}\left[\left(\sum_k (a_k'')^2\right)^{1/2} + \left(\sum_k (b_k'')^2\right)^{1/2}\right] \\
&< \infty. \qquad (27)
\end{aligned}
$$

where I've used the "discrete" version of the Cauchy-Schwarz inequality, $\sum_k x_k y_k \leq (\sum_k x_k^2)^{1/2}(\sum_k y_k^2)^{1/2}$ and the fact that $\sum_{k\geq 1} 1/k^2 = \pi^2/6$. Also, we made use of the fact that h'' is in $L^2(0, 2\pi)$ so that $\sum_k((a_k'')^2 + (b_k'')^2) < \infty$. This proves the claim.

Now we're in a position to use Theorem 1. We let the set of functions ϕ_k in the theorem correspond to the functions $a_k \sin(kx)$ or $b_k \cos(kx)$—it doesn't really matter how we choose the correspondence—and then note that the derivatives of the $\phi_k'(x)$ are either $a_k' \cos(kx)$ or $b_k' \sin(kx)$. In either case we obtain values for the M_k in Theorem 1 which are either $|a_k'|$ or $|b_k'|$. Thus from the bound (27) we have $\sum_k M_k < \infty$. We can conclude from this that the functions $h_n(x)$ converge uniformly to $h(x)$ as $n \to \infty$, which is just what Lemma 1 claims.

We can use this Lemma to show that if the Dirichlet data $h \in C^2([0, 2\pi])$ with $h(0) = h(2\pi), h'(0) = h'(2\pi)$ for the Laplacian then $\lim_{r\to 1^-} \|h(\theta) - v(r, \theta)\|_\infty = 0$ as follows. First, we know from Lemma 1 that $\lim_{n\to\infty} \|h - h_n\|_\infty = 0$. In precise ϵ-N terms, given any $\epsilon > 0$ we can find an N_1 such that

$$\|h - h_n\|_\infty < \epsilon/3 \qquad (28)$$

for all $n \geq N_1$. Let $u_n(r, \theta)$ denote the solution (15) truncated at n terms. Note that u_n has the same Fourier expansion as h, but with the coefficients multiplied by r^k. For any fixed $r < 1$ the very same argument as above shows that $\lim_{n\to\infty} \|u(r, \cdot) - u_n(r, \cdot)\|_\infty = 0$, or that given any $\epsilon > 0$ we can find an N_2 such that

$$\|u(r, \cdot) - u_n(r, \cdot)\|_\infty < \epsilon/3 \qquad (29)$$

for all $n \geq N_2$. Finally, we have

$$|h_n(\theta) - u_n(r, \theta)| = \left|\sum_{k=1}^{n} (1 - r^k)(a_k \sin(k\theta) + b_k \cos(k\theta))\right|$$

$$
\begin{aligned}
&\leq \sum_{k=1}^{n}(1-r^k)(|a_k|+|b_k|) \\
&\leq (1-r^n)\sum_{k=1}^{n}(|a_k|+|b_k|) \\
&= M(1-r^n) \qquad (30)
\end{aligned}
$$

where $M = \sum_{k=1}^{n}(|a_k| + |b_k|)$. We can choose r close enough to 1 so that $1 - r^n < \epsilon/3$ (specifically, take $r > (1-\epsilon/3)^{1/n}$). Then we obtain $h_n(\theta) - u_n(r,\theta) < \epsilon/3$ for all θ, i.e.,

$$\|h_n(\theta) - u_n(r,\theta)\|_\infty < \epsilon/3. \qquad (31)$$

Finally, note that inequalities (28), (29), and (31) yield, for $n \geq N = \max(N_1, N_2)$ and $r > (1-\epsilon/3)^{1/N}$,

$$
\begin{aligned}
\|u(r,\cdot) - h\|_\infty &\leq \|u(r,\cdot) - u_n(r,\cdot)\|_\infty + \|h_n(\theta) - u_n(r,\theta)\|_\infty + \|h - h_n\|_\infty \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 \\
&= \epsilon.
\end{aligned}
$$

In other words, for any given $\epsilon > 0$ if we choose r sufficiently close to 1, i.e., $1 > r > (1-\epsilon/3)^{1/N}$, we have $\|u(r,\cdot) - h\|_\infty < \epsilon$, which is precisely the statement that

$$\lim_{r \to 1^-} \|u(r,\cdot) - h\|_\infty = 0.$$

2.5 Poisson's Formula

It turns out that the Fourier series solution (15) can be written in another very compact way. Jam the formulas for the a_k and b_k directly into equation (15) to obtain

$$
\begin{aligned}
v(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} h(\alpha)d\alpha \\
&+ \frac{1}{\pi}\sum_{k=1}^{\infty} r^k \left[\sin(k\theta)\left(\int_0^{2\pi} h(\alpha)\sin(k\alpha)\,d\alpha\right) + \cos(k\theta)\left(\int_0^{2\pi} h(\alpha)\cos(k\alpha)\,d\alpha\right)\right] \\
&= \frac{1}{2\pi}\int_0^{2\pi} h(\alpha)d\alpha + \frac{1}{\pi}\sum_{k=1}^{\infty} r^k \left[\int_0^{2\pi} h(\alpha)\cos(k(\alpha-\theta))\,d\alpha\right] \\
&= \frac{1}{2\pi}\int_0^{2\pi} h(\alpha)\left[1 + 2\sum_{k=1}^{\infty} r^k \cos(k(\alpha-\theta))\right] d\alpha. \qquad (32)
\end{aligned}
$$

where we've used $\sin(k\theta)\sin(k\alpha)+\cos(k\theta)\cos(k\alpha)=\cos(k(\alpha-\theta))$.

You can actually sum the term in square brackets in equation (32), as follows. First, from $\cos(x)=\frac{e^{ix}+e^{-ix}}{2}$ we have

$$r^k\cos(k(\alpha-\theta)) = r^k\frac{e^{ik(\alpha-\theta)}+e^{-ik(\alpha-\theta)}}{2}.$$

Use the standard formula $x+x^2+x^3\cdots=\frac{x}{1-x}$ for the sum of a geometric series (if $|x|<1$) to find

$$\begin{aligned} 1+2\sum_{k=1}^{\infty} r^k\cos(k(\alpha-\theta)) &= 1+\sum_{k=1}^{\infty}(re^{i(\alpha-\theta)})^k+\sum_{k=1}^{\infty}(re^{-i(\alpha-\theta)})^k \\ &= 1+\frac{re^{i(\alpha-\theta)}}{1-re^{i(\alpha-\theta)}}+\frac{re^{-i(\alpha-\theta)}}{1-re^{-i(\alpha-\theta)}} \\ &= \frac{1-r^2}{1-2r\cos(\alpha-\theta)+r^2} \end{aligned}$$

after a bit of cancelling and algebra; note that since $0\le r<1$ the identity $x+x^2+x^3\cdots=\frac{x}{1-x}$ with $x=re^{\pm ik(\alpha-\theta)}$ is valid here. Equation (32) then becomes

$$v(r,\theta)=\frac{1-r^2}{2\pi}\int_0^{2\pi}\frac{h(\alpha)}{1-2r\cos(\alpha-\theta)+r^2}\,d\alpha, \tag{33}$$

the *Poisson integral formula.* Equation (33) gives us the value of the harmonic function at any point inside the ball in terms of the Dirichlet data h.

As a computational too though the Poisson formula is a bit difficult to use—you can't usually work the integral, except numerically.

3 Solvability on a General Domain

Solvability of Laplace's equation on a general domain, even in $\mathbb{R}^2$, isn't easy. There are a number of different approaches to the problem, some fairly old, some more modern, but there's a fair amount of abstract analysis involved no matter what approach you take. I'll only sketch the idea behind one common (and "modern", as in the last hundred years) approach.

Consider the problem of finding a harmonic function u on a bounded domain D in $\mathbb{R}^n$. We want $u=h$ on ∂D, where h is some given function;

let's suppose h is continuous, for simplicity. Let V denote the set of all functions ϕ defined on D with the properties that $\phi = h$ on ∂D and also that the quantity

$$Q(\phi) = \int_D |\nabla \phi|^2 \, dx < \infty.$$

The last condition dictates that ϕ not be too nasty inside D; in particular, $\nabla \phi$ should certainly exist. I claim that the minimizer of the functional Q is the harmonic function we seek.

To see this, note that Q is clearly bounded below by zero, and so the quantity

$$L = \inf_{\phi \in V} Q(\phi)$$

is well-defined. We can thus choose a sequence of functions $\phi_k \in V$ such that $Q(\phi_k) \to L$ as $k \to \infty$. Now here's the hard part, where some modern functional analysis is needed: We can "arrange" for this sequence ϕ_k to converge (in an appropriate sense) to some limit function in V, which I'll call u. We can show that $Q(u) = L$, so u is the minimizer of Q. I claim that u is the solution to the boundary value problem.

The proof that u is what we want is a calculus of variations problem. Consider perturbing u by a "small" function, to $u + \epsilon\eta$ where $\eta = 0$ on ∂D (so that $u + \epsilon\eta$ has the correct boundary values, and so is in V); let's also assume that η is C^1. Then since u is a minimizer we have

$$0 \leq Q(u + \epsilon\eta) - Q(u).$$

Write out the above inequality explicitly and simplify to obtain

$$0 \leq 2\epsilon \int_D \nabla \eta \cdot \nabla u \, dx + O(\epsilon^2).$$

The usual argument shows that for this to hold for all ϵ near zero we must have

$$\int_D \nabla \eta \cdot \nabla u \, dx = 0 \tag{34}$$

for all differentiable η. From the Divergence Theorem

$$\int_D \nabla \cdot (\eta \nabla u) \, dx = \int_D \nabla \eta \cdot \nabla u \, dx + \int_D \eta \triangle u \, dx = \int_{\partial D} \eta \frac{\partial u}{\partial \mathbf{n}} \, ds = 0 \tag{35}$$

since $\eta \equiv 0$ on ∂D. Equations (34) and (35) force

$$\int_D \eta \triangle u \, dx = 0$$

for all η, so that $\triangle u = 0$ in D. By construction $u = h$ on ∂D, and we're done!

Where in the above did I gloss over technical details? In several places (though the argument is correct). In order to force the sequence ϕ_k to converge, we have to let the set V contain functions for which $\nabla\phi$ exists in a rather generalized sense, specifically, as L^2 functions rather than continuous functions. But that raises the issue of whether the Divergence Theorem is still valid for these functions. Also, the statement that $\phi = h$ on ∂D must also be interpreted in a rather general sense, (a sort of L^2 agreement, rather than pointwise like you really want). And in order for all these arguments to make sense, we have to use an "improved" type of integration, specifically, Lebesgue integration. The Riemann stuff you saw in Calc II or reals isn't powerful enough to carry out these arguments!