Harmonic Functions

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1 Introduction

Let D be a bounded region in $\mathbb{R}^n$ (we'll focus on $\mathbb{R}^2$ for now) and $u(x,t)$ the temperature of D, where $x = (x_1, \ldots, x_n)$ and t is time. We found that (up to rescaling) u must satisfy the heat equation $u_t - \triangle u = 0$ in D, with appropriate boundary and initial conditions. For now we're going to study a special case of the heat equation, that in which the solution assumes a steady-state configuration, i.e., doesn't depend on time t. In this case $u_t \equiv 0$; the function u becomes a function only of the spatial variables and we find that $\triangle u = 0$ in D. This is called *Laplace's Equation*. The nonhomogeneous equation $\triangle u = f$ is called *Poisson's Equation*.

Definition: A function u which satisfies $\triangle u = 0$ is called *harmonic*.

Harmonic functions are a very special and important class of functions, not only in PDE, but also in complex analysis, electromagnetics, fluids, etc. But we can't solve $\triangle u = 0$ without more information. Specifically, the Dirichlet boundary condition from the heat equation is still needed, so we must specify $u = h$ on ∂D for some given function h. Alternatively we can specify $\frac{\partial u}{\partial \mathbf{n}} = g$ on ∂D for a given function g, though it turns out that g must satisfy a certain condition (more on this later) if a solution is to exist. The initial condition from the heat equation is no longer relevant, since u doesn't change with time now. All in all then our problem is to find a function which satisfies

$$\triangle u = 0 \text{ in } D, \quad (1)$$

$$u = h \text{ on } \partial D \quad (2)$$

OR

$$\frac{\partial u}{\partial n} = g \text{ on } \partial D. \quad (3)$$

Before proceeding it might be instructive to consider a few examples of harmonic functions. First, what does Poisson's equation looks like in one dimension? If D is the interval (a, b) then Poisson's equation just becomes

$$u''(x) = f(x) \text{ in } (a, b).$$

If have Dirichlet boundary conditions then $u(a) = h_1$ and $u(b) = h_2$, where h_1 and h_2 are some given numbers. This is easy to solve, just integrate twice and choose the constants of integration to get the right boundary values; it can always be done. In the special case that $f = 0$ the solution is a straight line. Harmonic functions in one dimension are just linear functions.

Exercise: Work out the solution to $u'' = f$ with $u(a) = h_1$, $u(b) = h_2$ in terms of h_1, h_2, a, b, and F, where $F'' = f$ is a "second" anti-derivative for f.

Now consider solving $u''(x) = 0$ on (a, b) with Neumann data (the special case $f \equiv 0$). The Neumann condition at $x = 0$ is $-u'(a) = g_1$ (we use $-u'(a)$, since the unit outward normal at $x = a$ points in the MINUS x direction) and $u'(b) = g_2$ for some given g_1, g_2. Integrate $u''(x) = 0$ once to find $u'(x) = m$ for some constant m. This immediately forces $g_1 = -m$ and $g_2 = m$ for otherwise we have a contradiction. If we do indeed have $g_1 = -g_2$ then we can take $m = g_2$, so $u'(x) = mx$, and integrating again shows that $u(x) = mx + c$ is harmonic and has the right Neumann data for ANY choice of c. In this case there is not a unique solution, but rather an entire family of solutions, all differing by an additive constant.

Exercise: Consider solving $u''(x) = f(x)$ on (a, b) with Neumann data $-u'(a) = g_1$, $u'(b) = g_2$. What relation must hold between g_1, g_2, and f in order for a solution to exist? Will it be unique?

In two dimensions it's easy to check that any linear function of the form $u(x, y) = ax + by + c$ for constants a, b, c, is harmonic. The function $u(x, y) = x^2 - y^2$ is also harmonic. Not surprisingly, there are infinitely many others.

2 Basic Properties of Laplace's Equation

Let's consider the big three questions for Laplace's equation: Existence, Uniqueness, and Stability. We'll also consider the allied question of "what interesting properties do harmonic functions have?" We are not yet in a position to answer all questions. We'll deal with uniqueness and stability first, but consider only special cases for existence. For the moment, let's focus on uniqueness.

2.1 Uniqueness

It's easy to show that there is only one function that can satisfy Poisson's equation with given Dirichlet boundary data. Suppose that both u_1 and u_2 are C^2 and satisfy Poisson's equation with forcing function f and boundary data h. Then the function $v = u_2 - u_1$ satisfies $\triangle v = 0$ in D and $v = 0$ on ∂D. We will show that $v = 0$ in D, so $u_1 = u_2$ in D.

Take the equation $\triangle v = 0$ and multiply both sides by v to obtain $v \triangle v = 0$. By Green's first identity

$$\int_D v \triangle v \, dV = \int_{\partial D} v \frac{\partial v}{\partial \mathbf{n}} \, dA - \int_D |\nabla v|^2 \, dV.$$

But since $v = 0$ on ∂D and since $\triangle v = 0$ this becomes

$$\int_D |\nabla v|^2 \, dV = 0. \tag{4}$$

But $|\nabla v|^2$ is clearly non-negative on D; if the integral is zero then we must have $\nabla v \equiv 0$ on D, so that v is constant. But since $v = 0$ on ∂D we conclude that $v \equiv 0$ throughout D. Thus $u_1 = u_2$ in D.

Now consider the same situation but with Neumann data. The same computations which led to equation (4) still work, and we conclude that $v = u_2 - u_1$ is constant. But since we now have only $\frac{\partial v}{\partial \mathbf{n}} = 0$ on ∂D, rather than $v = 0$, we can't conclude that $v \equiv 0$ on D, only that $v = c$, so that $u_2 = u_1 + c$. And indeed, it's easy to see that if $\triangle u_1 = f$ on D with $\frac{\partial u_1}{\partial n} = g$ then $u_2 = u_1 + c$ satisfies the same conditions for any choice of c.

So when dealing with Neumann boundary conditions we obtain uniqueness only up to an additive constant. We can obtain uniqueness by specifying one additional condition, e.g., the value of the solution u at a given point in D, or, for example, the condition that

$$\int_D u(x) \, dx = 0. \tag{5}$$

Thus, for example, if two solutions with the same Neumann data differ by a constant, say $u_2 = u_1 + c$, and both satisfy the additional condition (5) we can immediately deduce that $c = 0$, so $u_1 \equiv u_2$.

2.2 The Maximum Principle for Laplace's Equation

Let u be a solution to Laplace's equation $\triangle u = 0$ which is C^2 on D with u continuous up to and on the boundary of some region D (meaning that for any point $x_0 \in \partial D$ we have $\lim_{x \to x_0} u(x) = u(x_0)$, where x limits to x_0 from inside D). The maximum principle states that

$$\sup_{x \in D} u(x) = \sup_{x \in \partial D} u(x).$$

Note that the supremum might also be attained somewhere inside D too (consider the case in which u is constant)—the assertion is simply that it must be attained on the boundary.

Proof: The is exactly like the proof of the maximum principle for the heat equation. In fact, we use the same trick. Let $v(x) = u(x) + \epsilon |x|^2$, where $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ means the usual Pythagorean norm of x and ϵ is some small positive number. Then v has no maximum inside D, for if it did have a maximum at $x = p$ then

$$\triangle v(p) = \triangle u + 4\epsilon > 0$$

which is impossible at an interior maximum (from the second derivative test in Calculus III). We can conclude that the maximum value of v occurs on the boundary, so

$$\sup_{x \in D} v(x) = \sup_{x \in \partial D} v(x).$$

Suppose that p is a point in ∂D at which v attains its maximum value. Then we have, for $x \in D$,

$$u(x) < v(x) \leq v(p) = u(p) + \epsilon |p|^2 \leq \sup_{x \in \partial D} u(x) + \epsilon d^2$$

where d is the maximum distance from any point in D to the origin. But since ϵ is arbitrary we conclude that

$$u(x) \leq \sup_{x \in \partial D} u(x).$$

The same reasoning applied to $-u(x)$ proves that the minimum value is attained on the boundary. Also, the same reasoning proves the

Extended Maximum Principle: If $f \geq 0$ on D then a function u satisfying $\triangle u = f$ on D then u must attain its maximum value (but NOT necessarily minimum) value on ∂D. Similarly if $f \leq 0$ then u must attain its minimum (but not necessarily maximum) value on ∂D.

Actually, the maximum principle provides yet another proof of uniqueness. If u_1 and u_2 both satisfy Poisson's equation on D with the same f and Dirichlet data h then $v = u_2 - u_1$ is harmonic with zero boundary data. By the maximum principle the maximum and minimum value attained by v on D is attained on ∂D, and hence is zero. Thus v must be identically zero, so $u_1 = u_2$ on D.

2.3 Stability

Suppose that u_1 and u_2 both satisfy Poisson's equation with the same forcing function, so $\triangle u_j = f$ on D. Suppose that they have different boundary conditions, say $u_j = h_j$ on ∂D, for $j = 1, 2$. Then we can prove a stability result, namely

$$\sup_{x \in D} |u_2(x) - u_1(x)| \leq \sup_{x \in \partial D} |h_2(x) - h_1(x)|.$$

In other words, if the functions h_1 and h_2 are close in the supremum norm on ∂D then u_1 and u_2 are close in the supremum norm on all of D. The proof is easy. Let $v = u_2 - u_1$. Then v is harmonic and has boundary data $h_2 - h_1$. Define

$$M = \sup_{x \in \partial D} |h_2(x) - h_1(x)|$$

and note that $\sup_{x \in \partial D}(h_2(x) - h_1(x)) \leq M$ and $\inf_{x \in \partial D}(h_2(x) - h_1(x)) \geq -M$. By the maximum/minimum principle for harmonic functions

$$\sup_{x \in D} v(x) \leq \sup_{x \in \partial D} (h_2(x) - h_1(x)) \leq M, \tag{6}$$

$$\inf_{x \in D} v(x) \geq \inf_{x \in \partial D} (h_2(x) - h_1(x)) \geq -M, \tag{7}$$

Also note that $\sup_{x \in D} |v|$ equals either $\sup_{x \in D} v(x)$ or $-\inf_{x \in D} v(x)$. Equations (6) and (7) then yield

$$\sup_{x \in D} |v| \leq M.$$

Thus if M (the supremum norm of $h_2 - h_1$ on ∂D) is small then v has a small supremum norm, i.e., u_1 is close to u_2 in supremum norm.

2.4 The Green's Function and Green's Third Identity

Let $x = (x_1, x_2)$ denote a point in two dimensions and let $|x| = \sqrt{x_1^2 + x_2^2}$. The function

$$G(x) = \frac{1}{2\pi} \ln |x| \tag{8}$$

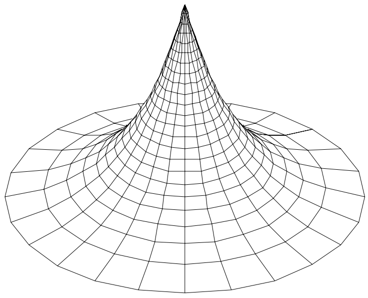
is called the *Green's Function* or *Green's Kernel* or *Fundamental Solution* for the Laplacian. In n dimensions with $n \geq 3$ the Green's function is

$$G(x) = -\frac{1}{\omega_n} |x|^{2-n} \tag{9}$$

where $x = (x_1, \ldots, x_n)$, $|x|$ is the magnitude of x, and ω_n is the surface area of the unit ball in n dimensions. The Green's kernel here plays a similar role as the Green's kernel for the heat equation. First, you can easily check that

$$\triangle G(x) = 0$$

IF $x \neq 0$. At $x = 0$ the function G has an asymptote, so $\triangle G$ makes no sense there. Here's a picture of $-G$ in two dimensions:



Note that $G(x)$ is radially symmetric about the singularity. Along any radial line from the singularity G drops off with distance r as $\frac{1}{2\pi}\ln(r)$.

Given a fixed point $y = (y_1, \ldots, y_n)$ in n dimensional space we can also consider the function $G(x - y)$, which translates G so that the singularity is at $x = y$. To understand the significance and usefulness of G we need *Green's Third Identity*, which we derive below for the two-dimensional case, though it works in any dimension.

Recall Green's second identity: Let D be a bounded domain in $\mathbb{R}^2$; for C^2 functions u and v we have

$$\int_D (u \bigtriangleup v - v \bigtriangleup u)\, dx = \int_{\partial D} (u\frac{\partial v}{\partial \mathbf{n}} - v\frac{\partial u}{\partial \mathbf{n}})\, ds$$

where $\mathbf{n}$ is an outward unit normal vector on ∂D (where $dx = dx_1\, dx_2$ and ds is arc length) Let's take y to be some fixed point inside D and then take $v(x) = G(x - y)$. Then $\bigtriangleup G = 0$ for $x \neq y$, where it's understood that we apply $\bigtriangleup$ in the x variable. We want to put $v = G$ into Green's second identity and see what comes out, but there's one problem: G is NOT C^2 on D, so Green's identity isn't valid. To get around this let us remove from D a tiny ball $B_\epsilon(y)$ of radius ϵ centered at y as illustrated below.

Let's use the notation D_ϵ for D with the ball $B_\epsilon(y)$ removed. On D_ϵ the function $G(x-y)$ is smooth and so Green's second identity is valid. If we put in $v(x) = G(x-y)$ (and u is still just some arbitrary C^2 function, not necessarily harmonic) then we obtain

$$-\int_{D_\epsilon} G(x-y) \triangle u(x)\, dx = \int_{\partial D_\epsilon} (u\frac{\partial G}{\partial \mathbf{n}} - G\frac{\partial u}{\partial \mathbf{n}})\, ds \qquad (10)$$

where all derivatives hitting G are with respect to the x variable. Now notice that ∂D_ϵ really consists of two pieces: ∂D and $\partial B_\epsilon(y)$. On ∂D the vector $\mathbf{n}$ points outward, while on $\partial B_\epsilon(y)$ the vector $\mathbf{n}$ points INTO the ball (which is OUT of D_ϵ). Let's take equation (10) and split the integrals over ∂D_ϵ into integrals over ∂D and $\partial B_\epsilon(y)$ to obtain

$$\begin{aligned} -\int_{D_\epsilon} G(x-y) \triangle u(x)\, dx &= \int_{\partial D} u\frac{\partial G}{\partial \mathbf{n}}\, ds - \int_{\partial D} G\frac{\partial u}{\partial \mathbf{n}}\, ds \\ &\quad + \int_{\partial B_{\epsilon(y)}} u\frac{\partial G}{\partial \mathbf{n}}\, ds - \int_{\partial B_{\epsilon(y)}} G\frac{\partial u}{\partial \mathbf{n}}\, ds. \qquad (11) \end{aligned}$$

Let's look at what happens to equation (11) as ϵ tends to zero.

Claim: As ϵ approaches zero the integral on the left in equation (11) becomes an integral over D. The last integral on the right vanishes, and the second to last integral approaches $-u(y)$. In summary, equation (11) becomes

$$-\int_{D} G(x-y) \triangle u(x)\, dx = \int_{\partial D} u\frac{\partial G}{\partial \mathbf{n}}\, ds - \int_{\partial D} G\frac{\partial u}{\partial \mathbf{n}}\, ds - u(y). \qquad (12)$$

Proof: We'll examine the left side first, then the right side. First of all, I claim that the left side tends to $-\int_D G(x-y) \triangle u(x)\, dx$. To see this, look at the difference

$$\int_D G(x-y) \triangle u(x)\, dx - \int_{D_\epsilon} G(x-y) \triangle u(x)\, dx = \int_{B_\epsilon(y)} G(x-y) \triangle u(x)\, dx. \tag{13}$$

If u is C^2 then $\triangle u(x)$ is bounded by some constant C near the point $x = y$, so the magnitude of the integral on the right above is bounded by

$$C \left| \int_{B_\epsilon(y)} G(x-y)\, dx \right|.$$

Also, $G(x-y)$ is just $\frac{1}{2\pi} \ln|x-y|$, and this integral can be worked explicitly. Just change variables—let $w = x - y$ so $dw = dx$. This integral becomes

$$C \int_{B_\epsilon(0)} G(w)\, dw.$$

Switch to polar coordinates and the integral becomes, explicitly

$$C \int_0^{2\pi} \int_0^\epsilon \frac{1}{2\pi} \ln(r) r\, dr\, d\theta = C \frac{\epsilon^2}{2} \left(\ln(\epsilon) - \frac{1}{2} \right)$$

which approaches zero as $\epsilon \to 0$. Thus the right side of equation (13) approaches zero, and so the integral over D_ϵ approaches the integral over D.

Now let's examine the right side of equation (11). First, the terms involving integral over ∂D don't even involve ϵ, so there's nothing to do there. The very last term involving $G \frac{\partial u}{\partial \mathbf{n}}$ tends to zero as ϵ approaches zero. To see this, note that if u is C^2 on D then $\frac{\partial u}{\partial \mathbf{n}} = \nabla u \cdot \mathbf{n}$ is bounded on $\partial B_{\epsilon(y)}$. Thus the final integral in equation (11) can be bounded by

$$C \int_{\partial B_{\epsilon(y)}} G(x-y)\, ds$$

where the integral is with respect to x. Parameterize the boundary of $B_\epsilon(y)$ as $x_1 = y_1 + \epsilon \cos(w)$, $x_2 = y_2 + \epsilon \sin(w)$ for $0 \leq w < 2\pi$. Also, $ds = \epsilon dv$ and then this integral becomes

$$C \int_0^{2\pi} \frac{1}{2\pi} \ln(\epsilon) \epsilon\, dw = C \epsilon \ln(\epsilon)$$

which approaches zero as $\epsilon \to 0$, as asserted.

Finally, let's examine the most interesting term in equation (11), the second to last integral on the right. First of all let's again parameterize the boundary of $B_\epsilon(y)$ as $x_1 = y_1 + \epsilon\cos(w)$, $x_2 = y_2 + \epsilon\sin(w)$ for $0 \le w < 2\pi$. Then $ds = \epsilon dw$ and an outward unit normal vector is just $\mathbf{n} = (\cos(w), \sin(w))$. It's easy to check that

$$\nabla_x G(x, y) = \left(-\frac{x_1 - y_1}{2\pi r^2}, -\frac{x_2 - y_2}{2\pi r^2}\right)$$

where $r^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2$ and ∇_x means compute the gradient with respect to the x variable. Now compute $\frac{\partial G}{\partial \mathbf{n}} = \nabla_x G \cdot \mathbf{n}$, making sure to put everything in terms of w. We obtain

$$\frac{\partial G}{\partial \mathbf{n}} = -\frac{1}{2\pi\epsilon}$$

on $\partial B_{\epsilon(y)}$. Thus

$$\int_{\partial B_{\epsilon(y)}} u\frac{\partial G}{\partial \mathbf{n}}\, ds = -\int_0^{2\pi} \frac{1}{2\pi\epsilon} u(y + \epsilon(\cos(w), \sin(w)))\, dw. \tag{14}$$

But since u is continuous, if ϵ is small then $u(y+\epsilon(\cos(w), \sin(w)))$ is close to $u(y)$ (which is constant in x). In the limit that ϵ approaches zero the integral on the right in equation (14) looks like

$$-\frac{1}{2\pi\epsilon} u(y) \int_0^{2\pi} dw = -u(y).$$

This proves that the second to last integral on the right in equation (11) approaches $u(y)$, as asserted, and finishes the proof of the claim.

In summary, we can write equation (12) as

$$u(y) = \int_D G(x-y)\triangle u(x)\, dx + \int_{\partial D} u(x)\frac{\partial G}{\partial \mathbf{n}}(x-y)\, ds_x - \int_{\partial D} G(x-y)\frac{\partial u}{\partial \mathbf{n}}(x)\, ds_x \tag{15}$$

where ds_x means integrate with respect to x. This is *Green's Third Identity.*

As a special case, suppose we have a harmonic function u on D, so $\triangle u = 0$ in D, with $u = h$ on ∂D and $\frac{\partial u}{\partial \mathbf{n}} = g$ on ∂D. Then Green's third identity tells us how to find u at any interior point y, as

$$u(y) = \int_{\partial D} \frac{\partial G}{\partial \mathbf{n}}(x - y)h(x)\, ds_x - \int_{\partial D} G(x - y)g(x)\, ds_x. \tag{16}$$

But be warned—if we simply make up functions g and h on ∂D and plug them into equation (16) we won't obtain a harmonic function u with Dirichlet data h and normal derivative g, for we already know that h alone determines u, or likewise, g alone determines u up to an additive constant. We can't pick BOTH of them and expect to find a solution. If you do choose both and plug into equation (16) you'll end up with a function that is harmonic on D, but it probably won't have the correct Dirichlet OR Neumann data.

Exactly the same argument given above works in higher dimensions too, though I won't go through it in detail. But equations (15) and (16) are true in $\mathbb{R}^n$.

2.5 The Mean Value Property

This is without doubt the most amazing property of harmonic functions. In fact, it's a property that ONLY harmonic functions have. Suppose that $\triangle u = 0$ on some region D. Let B be a ball of radius r around some point $y = (y_1, y_2)$ in D (r not necessarily small, but small enough so B is contained in D).

Claim: The value of u at the center of the ball equals the average value of u over the boundary of the ball, i.e.,

$$u(y) = \frac{1}{|\partial B|}\int_{\partial B} u\,ds = \frac{1}{2\pi}\int_0^{2\pi} u(y_1 + r\cos(w), y_2 + r\sin(w))\,dw \qquad (17)$$

where $|\partial B| = 2\pi r$ denotes the length of the boundary of B (or in $n \geq 3$ dimensions the surface "area" of the boundary).

To prove the Mean Value Property, start with Green's third identity in the form (16). We have

$$u(y) = \int_{\partial B} u(x)\frac{\partial G}{\partial \mathbf{n}}(x-y)\,ds_x. - \int_{\partial B} G(x-y)\frac{\partial u}{\partial \mathbf{n}}(x)\,ds_x.$$

Since B is a ball of radius r we'll parameterize its boundary as $x_1 = y_1 + r\cos(w)$, $x_2 = y_2 + r\sin(w)$, so $ds_x = r\,dw$. On ∂B the function $G(x-y)$ becomes just a constant $\frac{1}{2\pi}\ln(r)$, while $\frac{\partial G}{\partial \mathbf{n}}$ also becomes a constant, $\frac{1}{2\pi r}$. The above equation becomes

$$u(y) = \frac{1}{2\pi r}\int_0^{2\pi} u(y_1 + r\cos(w), y_2 + r\sin(w))r\,dw + \frac{\ln(r)}{2\pi}\int_{\partial B}\frac{\partial u}{\partial \mathbf{n}}\,ds_x \quad (18)$$

The second integral above is actually zero. To see this note that if u is harmonic then from Green's FIRST identity with $v(x) \equiv 1$ we have

$$\int_{\partial B} \frac{\partial u}{\partial \mathbf{n}}\, ds = \int_B \triangle u\, dx = 0.$$

Plugging this into equation (18) proves the claim.

Again, the same argument works in $\mathbb{R}^n$ for $n > 2$. The mean value property even holds in one dimension, where it becomes the assertion that if $u'' = 0$ then

$$u(x) = \frac{1}{2}(u(x-r) + u(x+r))$$

which is certainly true for harmonic (i.e., linear) functions.

The mean value property also works if you use the interior of the ball, instead of the boundary.

Exercise: Prove the last claim. Specifically, if u has the mean value property in the form (17) then

$$u(y) = \frac{1}{|B|} \int_B u\, dA$$

where $|B| = 4\pi r^2$ is the area of the ball B and dA means $dx_1\, dx_2$. Then prove the converse; assume whatever continuity or differentiability you need from u.

The mean value property provides another proof of the maximum principle. Informally, if x_0 is an interior point for a region D then we can put a small ball of some radius around x_0. Now we can't have $u(x_0) > u(x)$ for all $x \in D$, because $u(x_0)$ is an average of the values of u on the boundary of any ball surrounding x_0. But the average can never be larger than EVERY point out of which it is constructed.

It turns out that harmonic functions are the ONLY ones which possess the mean value property. This is easy to prove, at least if you're willing to restrict your attention to C^2 functions. For the moment, let u be ANY C^2 function. Choose a fixed point y in D and let B_r denote a ball of radius r around y (r small enough so B_r is contained in D). Define a function $\phi(r)$ as

$$\phi(r) = \frac{1}{|\partial B_r|} \int_{\partial B_r} u\, ds = \frac{1}{2\pi} \int_0^{2\pi} u(y_1 + r\cos(t), y_2 + r\sin(t))\, dt.$$

The function $\phi(r)$ is just the average value of u over the ball of radius r centered at y. Compute

$$\begin{aligned}
\phi'(r) &= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial u}{\partial x_1}(y_1 + r\cos(t), y_2 + r\sin(t))\cos(t)\right. \\
&+ \left.\frac{\partial u}{\partial x_2}(y_1 + r\cos(t), y_2 + r\sin(t))\sin(t)\right)\,dt \\
&= \frac{1}{|\partial B_r|}\int_{\partial B_r}\frac{\partial u}{\partial \mathbf{n}}\,ds \\
&= \frac{1}{|\partial B_r|}\int_{B_r}\triangle u\,dA. \qquad (19)
\end{aligned}$$

The last step is Green's first identity. Equation (19) holds for any C^2 function. Note also that $\lim_{r\to 0^+}\phi(r) = u(y)$ if u is continuous, that is, any continuous function has the mean value property on a ball of radius zero.

Now suppose that u is some function which is NOT harmonic. Then we can find some ball B_R contained in D with either $\triangle u > 0$ or $\triangle u < 0$ on B_R; assume the former. For $r < R$ equation (19) forces $\phi'(r) > 0$. Since $\lim_{r\to 0^+}\phi(r) = u(y)$ this shows that $u(y) < \phi(r)$ for any $r > 0$, that is, u does not have the mean value property. A similar conclusion holds if $\triangle u < 0$ anywhere. The proves that if u has the mean value property then u is harmonic.

By the way, this also gives another proof of the mean value property for harmonic functions: If $\triangle u = 0$ then $\phi'(r) = 0$, i.e., ϕ is constant. Since $\lim_{r\to 0^+}\phi(r) = u(y)$, u must have the mean value property.