The Heat Equation on a Bounded Domain

Uniqueness and Stability

MA 436

0.1 Introduction

We've solved the heat (and wave) equations in one space variable on the whole real line (and the half-line). For the next couple weeks we're going to concentrate on solving them on a bounded interval, starting with the heat equation. We want to understand existence, uniqueness, and stability for the problem of finding a function $u(x,t)$ which satisfies

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= 0, \\ u(x,0) &= f(x) \end{aligned} \tag{1}$$

for $a < x < b$ and $t > 0$, where $f(x)$ is some given initial temperature. As we've seen, we also need boundary conditions. At the left end $x = a$ we can take a Dirichlet condition of the form $u(a,t) = 0$, or more generally, $u(a,t) = h(t)$ for some specified function $h(t)$; this corresponds to knowing or controlling the temperature at the left end of the bar. Alternatively, we can take a Neumann boundary condition of the form $u_x(a,t) = g(t)$ for some specified function $g(t)$ ($g \equiv 0$ is common); this corresponds to knowing or controlling the rate at which heat energy enters or leaves the left end of the bar. We can make a similar choice or boundary conditions at $x = b$. There are also many other boundary conditions possible; we'll look some later.

As it turns out proving the *existence* of solutions to the heat equation on a bounded domain is a bit difficult. We'll start by examining the uniqueness and stability properties.

0.2 The Maximum Principle

The maximum principle is a remarkable property of the heat equation (other PDE's also have such a property; it usually turns out to be very useful). For simplicity we'll restrict our attention to the interval $0 \leq x \leq 1$, although everything goes through on any other interval. Let R be the rectangle $0 \leq x \leq 1$, $0 \leq t \leq T$, let B denote the union of the sides $t = 0$, $x = 0$,and $x = 1$ (all but the top), and let M denote the maximum value of u over B. Refer to the figure below.

The Maximum Principle: *If $u(x,t)$ satisfies the heat equation for $0 < x < 1$ and $0 < t < T$ then the maximum value of u occurs at $t = 0$ (at the initial condition) or for $x = 0$ or $x = 1$ (at the ends of the rod). More precisely,*

$$\sup_R u(x,t) = \sup_B u(x,t).$$

Note that the boundary conditions aren't even mentioned—it's true regardless. Also, the maximum principle doesn't preclude u also attaining the maximum value away from B; it's just that the maximum must also be attained on B.

The basic idea of the proof is from calculus, with a slight subtlety. At an interior maximum (x_0, t_0) the first derivative $u_t(x_0, t_0) = 0$. Also, if (x_0, t_0) is a maximum then we must have $u_{xx}(x_0, t_0) \leq 0$ (since if $u_{xx}(x_0, t_0) > 0$ then the function $u(x, t_0)$ of x would be concave up near $x = x_0$, and so couldn't have a maximum at $x = x_0$). So we must have $u_{xx}(x_0, t_0) \leq 0$. A strict inequality $u_{xx}(x_0, t_0) < 0$ is easy to rule out, for given that $u_t(x_0, t_0) = 0$ we'd then have $u_t - u_{xx} > 0$ at (x_0, t_0), contradicting the fact that u satisfies the heat equation. But the possibility that $u_t(x_0, t_0) = 0$ isn't so easy to rule out.

So we use a silly little trick to get around this. Define a new function $v(x,t)$ as

$$v(x,t) = u(x,t) + \epsilon x^2$$

where $\epsilon > 0$. You can check that $v_t - v_{xx} = -2\epsilon < 0$.

The first claim is that $v(x,t)$ must attain its maximum value on R on one of the sides $t = 0$, $x = 0$, or $x = 1$. To see this, first note that the maximum cannot be inside R, for at that point we'd have $v_t = 0$ and $v_{xx} \leq 0$ (just as for u above), violating $v_t - v_{xx} = -2\epsilon < 0$ (note that the strict inequality $\epsilon > 0$ saves us here). So the maximum for v occurs on one of the four sides, $x = 0, x = 1, t = 0$, or $t = T$. But it can't occur on the top side $t = T$: suppose it does occur there, at some point (x_0, T). Then again we'd have $v_{xx}(x_0, T) \leq 0$, and also $v_t(x_0, T) \geq 0$, for

$$v_t(x_0, T) = \lim_{h \to 0^+} \frac{v(x_0, T) - v(x_0, T-h)}{h} \geq 0$$

since by assumption $v(x_0, T) \geq v(x_0, T - h)$. But this would imply that $v_t(x_0, T) - v_{xx}(x_0, T) \geq 0$, a contradiction to $v_t - v_{xx} = -2\epsilon < 0$. Since the maximum can't occur on $t = T$ it must occur on B, that is,

$$\sup_R v(x,t) \leq \sup_B v(x,t) \tag{2}$$

Now from $v(x,t) = u(x,t) + \epsilon x^2$ we have

$$\sup_B v(x,t) \leq \sup_B u(x,t) + \sup_B \epsilon x^2 = M + \epsilon. \tag{3}$$

From $u(x,t) = v(x,t) - \epsilon x^2$ and $0 \leq x \leq 1$ it's obvious that

$$\sup_R u(x,t) \leq \sup_R v(x,t). \tag{4}$$

Stringing together inequalities (4), (2), and (3) (in that order) shows that

$$\sup_R u(x,t) \leq M + \epsilon \tag{5}$$

for *any* $\epsilon > 0$, forcing $\sup_R u \leq M$. Of course since u is continuous we have to actually have equality, so $\sup_R u = M$. This proves the maximum principle.

By considering the function $-u(x,t)$, the same argument shows that the minimum must also occur on the same portion of the boundary.

0.3 Uniqueness and Stability

The maximum principle makes uniqueness and stability easy. We showed that if two solutions to the heat equation have initial conditions which are close in the L^2 norm then the solutions at any time are close in L^2. This is also true in the supremum norm. Suppose that $u_1(x,t)$ and $u_2(x,t)$ are solutions to the heat equation on $(0,1)$, both with boundary conditions $u_1(0,t) =$

$u_2(0,t) = a(t)$ and $u_2(1,t) = u_2(1,t) = b(t)$ for some functions $a(t)$ and $b(t)$. Suppose the initial conditions are $u_1(x,0) = f_1(x)$ and $u_2(x,0) = f_2(x)$, and let $M = \sup_{0<x<1} |f_2(x) - f_1(x)|$. Then $\sup_{0<x<1} |u_1(x,t) - u_2(x,t)| \leq M$ for all $t > 0$. In other words, if u_1 and u_2 start with close initial conditions, they'll stay close for all time. The proof is easy. Let $v = u_2 - u_1$. Then v satisfies the heat equation with zero boundary conditions and initial condition $f_2(x) - f_1(x)$. By the maximum principle the maximum (and minimum) value of v occurs on $t = 0$, $x = 0$, or $x = 1$. But on $x = 0$ and $x = 1$ the function v is identically zero. So the maximum and minimum values of v are either zero, or occur when $t = 0$. Either way, $|v(x,t)|$ cannot exceed $\sup_{0<x<1} |v(x,0)| = \sup_{0<x<1} |f_2(x) - f_1(x)|$.

Of course, this immediately implies that the heat equation has a unique solution for given boundary conditions and initial data. If two solutions agree on the three sides $t = 0$, $x = 0$, $x = 1$ then the maximum absolute value of their difference over R is zero, i.e., they're the same everywhere.