

Green's Functions and the Heat Equation

MA 436

Kurt Bryan

0.1 Introduction

Our goal is to solve the heat equation on the whole real line, with given initial data. Specifically, we seek a function $u(x, t)$ which satisfies

$$\begin{aligned}\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= 0, \\ u(x, 0) &= f(x)\end{aligned}\tag{1}$$

for $-\infty < x < \infty$ and $t > 0$, where $f(x)$ is some given initial temperature. We'll make assumptions about f later. Of course we want existence, uniqueness, and stability results.

Recall that the solution to the wave equation on the whole real line was given by the D'Alembert formula, which constructed the solution via various integrals involving the initial data. The solution to the heat equation (1) on the whole real line is also given by such formula,

$$u(x, t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} f(y) dy.\tag{2}$$

But of course you're wondering "where in the world does that come from?" That's what we should look at.

0.2 Mollification

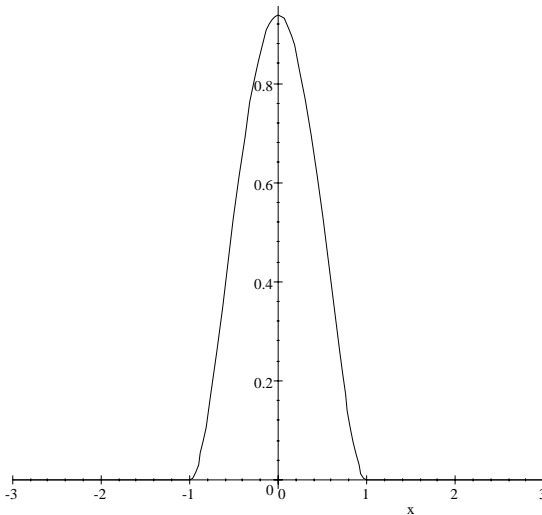
Suppose that $\phi(x)$ is a reasonably smooth (say C^1 or better) non-negative function defined for all real x which decays to zero at infinity, with

$$\int_{-\infty}^{\infty} \phi(x) dx = 1.$$

To help you visualize a specific example, we can use the function

$$\phi(x) = \begin{cases} 0, & x \leq -1, \\ \frac{15}{16}(x-1)^2(x+1)^2, & -1 < x < 1, \\ 0, & x \geq 1 \end{cases}$$

although this is just an example. Nothing that follows depends on this particular choice. In this case ϕ looks like



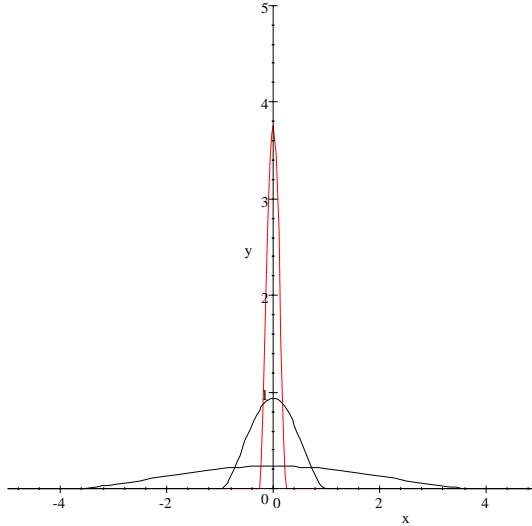
Now define the function

$$\psi(x, t) = \frac{1}{t} \phi\left(\frac{x}{t}\right).$$

You can easily check that for any $t > 0$ we have

$$\begin{aligned} \int_{-\infty}^{\infty} \psi(x, t) dx &= \int_{-\infty}^{\infty} \frac{1}{t} \phi\left(\frac{x}{t}\right) dx, \\ &= \int_{-\infty}^{\infty} \phi(u) du, \\ &= 1, \end{aligned} \tag{3}$$

where I used the substitution $u = x/t$ to get the last integral. What does the function $\psi(x, t)$ look like (as a function of x) for various choices of $t > 0$? Well, if $t = 1$ then obviously $\psi(x, 1) = \phi(x)$, so it looks just like the figure above. If t is large then the $1/t$ factor in front of ϕ tends to squash the function down toward zero, while the x/t factor inside expands the base of ϕ . Conversely, if t is small then the $1/t$ factor increases the height of the function and the base shrinks (think it through carefully). Here are $\psi(x, 1)$, $\psi(x, 4)$, and $\psi(x, 1/4)$ all plotted together:



Recall, however, in each case the area under the curve is exactly one.

Suppose that $w(x)$ is a bounded and continuous function on the real line, say bounded by some constant M . Consider the expression

$$\int_{-\infty}^{\infty} \psi(y, t)w(y) dy$$

as a function of t . We want to examine the behavior of this integral as a function of t . In particular, what limit if any does this integral approach as t approaches zero?

Here's the “informal” version of the argument: For small positive t the function $\psi(y, t)$ is “concentrated” near $y = 0$. It's not hard to see that the product $\psi(y, t)w(y)$ is also concentrated near $y = 0$. But if y is close to zero then $w(y) \approx w(0)$, with the approximation improving for y closer to zero, so if t is really close to zero then $\psi(y, t)w(y) \approx \psi(y, t)w(0)$. In this case

$$\int_{-\infty}^{\infty} \psi(y, t)w(y) dy \approx \int_{-\infty}^{\infty} \psi(y, t)w(0) dy = w(0)$$

since $\int_{-\infty}^{\infty} \psi(y, t) dy = 1$. In short,

$$\lim_{t \rightarrow 0} \int_{-\infty}^{\infty} \psi(y, t)w(y) dy = w(0) \tag{4}$$

if w is continuous. You might recognize that as t gets close to zero the function $\psi(x, t)$ looks like a “Dirac delta” function.

Here's the hardcore version of the "informal" argument (if you've never done ϵ/δ proofs, feel free to skip this paragraph). Choose a number $\epsilon > 0$. Since w is continuous, for any such $\epsilon > 0$ there is some $\delta > 0$ (dependent on ϵ) such that $|w(x) - w(0)| < \epsilon$ for all $|x| < \delta$. Now note that from equation (3) and the fact that $w(0)$ is a constant we have

$$w(0) = \frac{1}{t} \int_{-\infty}^{\infty} \phi(y/t) w(0) dy$$

Then

$$\begin{aligned} \frac{1}{t} \int_{-\infty}^{\infty} \phi(y/t) w(y) dy - w(0) &= \frac{1}{t} \int_{-\infty}^{\infty} \phi(y/t) (w(y) - w(0)) dy \\ &= \frac{1}{t} \int_{-\delta}^{\delta} \phi(y/t) (w(y) - w(0)) dy \\ &\quad + \frac{1}{t} \int_{-\infty}^{-\delta} \phi(y/t) (w(y) - w(0)) dy \\ &\quad + \frac{1}{t} \int_{\delta}^{\infty} \phi(y/t) (w(y) - w(0)) dy \end{aligned} \quad (5)$$

(I split the integral into three ranges). Take absolute values of both sides of equation (5) and use the triangle inequality to conclude that

$$\begin{aligned} \left| \frac{1}{t} \int_{-\infty}^{\infty} \phi(y/t) w(y) dy - w(0) \right| &\leq \left| \frac{1}{t} \int_{-\delta}^{\delta} \phi(y/t) (w(y) - w(0)) dy \right| \\ &\quad + \left| \frac{1}{t} \int_{-\infty}^{-\delta} \phi(y/t) (w(y) - w(0)) dy \right| \\ &\quad + \left| \frac{1}{t} \int_{\delta}^{\infty} \phi(y/t) (w(y) - w(0)) dy \right| \end{aligned} \quad (6)$$

The first integral on the right above can be bounded by noting that $|w(y) - w(0)| < \epsilon$ for $|y| < \delta$, so

$$\left| \frac{1}{t} \int_{-\delta}^{\delta} \phi(y/t) (w(y) - w(0)) dy \right| \leq \epsilon \left| \frac{1}{t} \int_{-\delta}^{\delta} |\phi(y/t)| dy \right| \leq \epsilon \frac{1}{t} \int_{-\infty}^{\infty} \phi(y/t) dy = \epsilon. \quad (7)$$

(note $|\phi| = \phi$). The other two integrals on the right in equation (6) are similar to each other. For example, since $|w(x)| \leq M$ for all x , $|w(y) - w(0)| \leq 2M$ and so

$$\left| \frac{1}{t} \int_{\delta}^{\infty} \phi(y/t) (w(y) - w(0)) dy \right| \leq 2M \frac{1}{t} \int_{\delta}^{\infty} \phi(y/t) dy.$$

Change variables in the last integral (use $x = y/t$, so $y = xt$, $dy = t dx$) and find

$$\left| \frac{1}{t} \int_{\delta}^{\infty} \phi(y/t)(w(y) - w(0)) dy \right| \leq 2M \int_{\delta/t}^{\infty} \phi(x) dx. \quad (8)$$

A similar bound hold for the integral from $-\infty$ to $-\delta$. Using the bounds of equation (7) and (8) in inequality (6) produces

$$\left| \frac{1}{t} \int_{-\infty}^{\infty} \phi(y/t)w(y) dy - w(0) \right| \leq \epsilon + 2M \int_{\delta/t}^{\infty} \phi(x) dx + 2M \int_{-\infty}^{-\delta/t} \phi(x) dx.$$

Take the limit of both sides above for $t \rightarrow 0^+$. Both integrals limit to zero (think about why), leaving us with

$$\lim_{t \rightarrow 0} \left| \frac{1}{t} \int_{-\infty}^{\infty} \phi(y/t)w(y) dy - w(0) \right| \leq \epsilon.$$

But $\epsilon > 0$ is arbitrary, so the left side above is zero and we have proved equation (4).

Now let's look at what happens to the integral

$$\int_{-\infty}^{\infty} \psi(y - x, t)w(y) dy$$

for small t . In this case the function $\psi(y - x, t)$ is just $\psi(y, t)$ translated x units to the right, and so for small t the function $\psi(y, t)$ is concentrated near $y = x$. A simple change of variables shows that

$$\lim_{t \rightarrow 0} \int_{-\infty}^{\infty} \psi(y - x, t)w(y) dy = w(x). \quad (9)$$

We'll call a smooth function $\psi(x, t)$ with the property (9) a *mollifier*. The verb "to mollify" means "to smooth out", and that's what $\psi(x, t)$ does. Here's an example. Let $w(x)$ be defined as

$$w(x) = \begin{cases} 0, & x \leq -1, \\ x + 1, & -1 < x \leq 0, \\ 1, & 0 < x < 1, \\ 2, & 1 \leq x < 3, \\ 0, & x > 3 \end{cases}$$

Here's what $w(x)$ looks like:

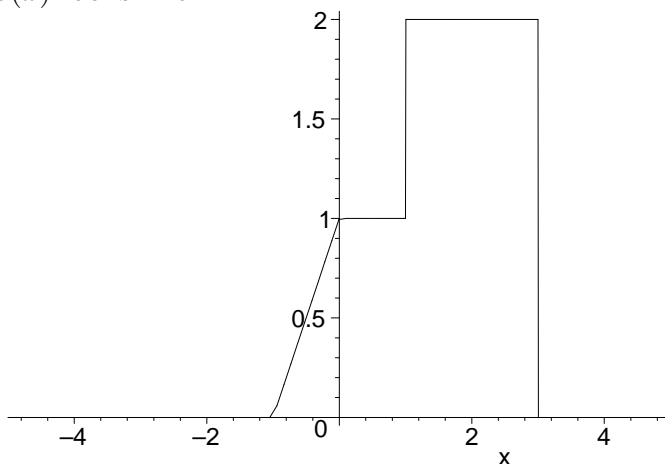


Figure 1: The Function $w(x)$

Note that w is discontinuous. Define the function

$$W(x, t) = \int_{-\infty}^{\infty} \psi(y - x, t)w(y) dy \quad (10)$$

where $\psi(y, t) = \frac{1}{t}\phi(y/t)$ for the function ϕ defined earlier. We know that $\lim_{t \rightarrow 0} W(x, t) = w(x)$, but it's instructive to plot the function $W(x, t)$ for some non-zero values of t . For example, when $t = 1$ here is what $W(x, 1)$ looks like compared to $w(x)$:

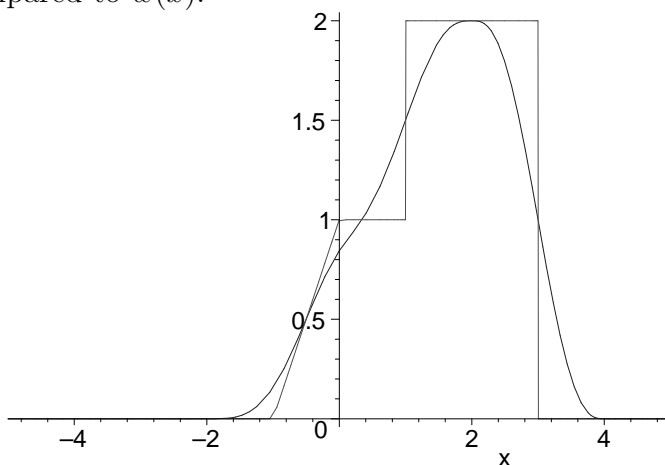


Figure 2: The Function $W(x, 1)$

It's a smoothed out version of w ! If you look at the definition of $W(x, t)$

you see that the integral simply takes the values of $w(y)$ near $y = x$ and averages them, weighted by the function $\psi(y - x, t)$. The bigger the value of t , the broader the range for the average, and the smoother W becomes. For small values of t the average is over a tight range about $y = x$. Since w doesn't change much over such a small range, the average near any point $y = x$ is about equal to $w(x)$. Here's a plot of $W(x, 4)$. It's pretty smooth.

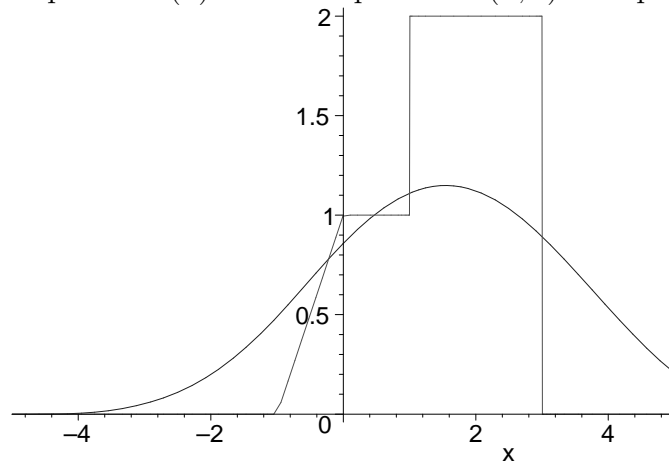


Figure 3: The Function $W(x, 4)$

And finally, here's $W(x, 1/4)$. It's a much better approximation to $w(x)$.

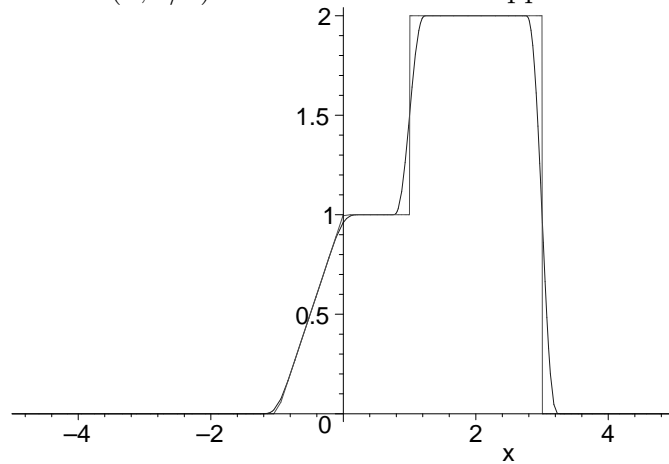


Figure 4: The Function $W(x, 1/4)$

A note on smoothness. Let us apply $\frac{\partial}{\partial x}$ to both sides of equation (10).

We obtain

$$\frac{\partial W}{\partial x} = \int_{-\infty}^{\infty} -\frac{\partial \psi}{\partial x}(y-x, t)w(y) dy$$

where the derivative has been slipped inside the integral. This is permissible if $w(y)$ and/or $\frac{\partial \psi}{\partial x}$ decay fast enough at infinity (certainly if either has compact support). Clearly it also requires that ψ be differentiable in x . The point here is that if ψ is differentiable then W is differentiable, even if w is not. Repeating the process shows that

$$\frac{\partial^n W}{\partial x^n} = \int_{-\infty}^{\infty} (-1)^n \frac{\partial^n \psi}{\partial x^n}(y-x, t)w(y) dy$$

again, provided that $\frac{\partial^n \psi}{\partial x^n}$ exists. The mollified function $W(x, t)$ will be just as smooth as ψ , and as t approaches zero W approaches w .

Mollification is an important process in PDE. Certain things can't be done to discontinuous or non-smooth functions (like differentiate them). Mollification allows you to replace non-smooth functions with smooth approximations. You then operate on these smooth approximations, take limits as t goes to zero, and hope that what you did to the smooth approximation makes sense for the original non-smooth function.

Also, a function $\psi(x, t)$ can be a mollifier without being of the form $\frac{1}{t} \phi(\frac{x}{t})$. For example, we can take

$$\psi(x, t) = \frac{1}{t^\alpha} \phi\left(\frac{x}{t^\alpha}\right). \tag{11}$$

where ϕ is some smooth function and $\alpha > 0$. You can easily verify that the same arguments above apply here—as t gets small, $\psi(x, t)$ gets “concentrated” near $x = 0$.

The reason we're considering mollification is this: Look back at the function $w(x)$ in Figure 1, and suppose that w is the initial temperature data for some bar at time $t = 0$. As t increases we'd expect that the heat diffuses and “smears out”, just as the function $W(x, t)$ does as t increases in Figures 2 to 4. In fact the diffusion of heat is a mollification process, if you choose the function ψ correctly.

0.3 Solving the Heat Equation

Suppose that $\psi(x, t)$ is a mollifying function, that is, $\psi(x, t)$ is defined for $-\infty < x < \infty$ and $t > 0$, and

$$\lim_{t \rightarrow 0^+} \int_{-\infty}^{\infty} \psi(y - x, t) f(y) dy = f(x)$$

for any continuous function f . To find a solution $u(x, t)$ to the heat equation with $u(x, 0) = f(x)$ we'll define

$$u(x, t) = \int_{-\infty}^{\infty} \psi(y - x, t) f(y) dy \quad (12)$$

for $t > 0$ for some suitable mollifier ψ . Then $u(x, 0) = f(x)$, in the sense that

$$\lim_{t \rightarrow 0^+} u(x, t) = f(x). \quad (13)$$

We'll sometimes have to be careful in writing $u(x, 0) = f(x)$, since plugging $t = 0$ into the integral of equation (12) might not make sense, but the limit as $t \rightarrow 0^+$ will. So technically, the initial condition is only satisfied in the sense of equation (13).

Now suppose that the mollifier $\psi(x, t)$ satisfies the heat equation, so $\psi_t - \psi_{xx} = 0$. Then the function $u(x, t)$ defined by equation (12) will also satisfy the heat equation, for

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) \int_{-\infty}^{\infty} \psi(y - x, t) f(y) dy, \\ &= \int_{-\infty}^{\infty} (\psi_t(y - x, t) - \psi_{xx}(y - x, t)) f(y) dy, \\ &= 0. \end{aligned}$$

since $\psi_t - \psi_{xx} = 0$. There is also the issue of slipping the derivatives inside the integral, but we can do this if ψ and its derivatives decay rapidly at infinity, or if f decays rapidly.

Solving the heat equation thus comes down to finding a mollifying function $\psi(x, t)$ which satisfies the heat equation. Equation (11) provides a means for constructing mollifiers, so let's see if we can find one that satisfies the heat equation. A good start would be to try $\psi(x, t)$ of the form

$$\psi(x, t) = \frac{1}{t^\alpha} \phi\left(\frac{x}{t^\alpha}\right)$$

for some function ϕ . We have to choose ϕ appropriately, and α also, to make this work in the heat equation. If we plug ψ into the heat equation we obtain

$$\psi_t - \psi_{xx} = -t^{-3\alpha}\phi''(x/t^\alpha) - \alpha xt^{-2\alpha-1}\phi'(x/t^\alpha) - \alpha t^{-\alpha-1}\phi(x/t^\alpha) = 0. \quad (14)$$

Note that the argument to ϕ is x/t^α . Let's get rid of that peculiarity. Define a new function η as $\eta(x) = \phi(x/t^\alpha)$ (note that η depends implicitly on t). Then

$$\begin{aligned} \frac{d\eta}{dx}(x) &= t^{-\alpha}\phi'(x/t^\alpha), \\ \frac{d^2\eta}{dx^2}(x) &= t^{-2\alpha}\phi''(x/t^\alpha). \end{aligned}$$

Use these two equations, and $\eta(x) = \phi(x/t^\alpha)$, replace ϕ and its derivatives in equation (14) with the equivalent in η . You obtain, after a bit of simplifying algebra,

$$\eta''(x) + \frac{\alpha x}{t}\eta'(x) + \frac{\alpha}{t}\eta(x) = 0. \quad (15)$$

This DE is actually pretty easy to solve. Write it in the form

$$\frac{\alpha x}{t}\eta'(x) + \frac{\alpha}{t}\eta(x) = -\eta''(x)$$

then notice that the left and right sides are both exact derivatives. In fact, the left side is $\frac{\alpha}{t}\frac{d}{dx}(x\eta(x))$, while the right is obviously $\frac{d\eta'}{dx}$. Integrate both sides and toss in a constant c_1 to find that

$$\frac{\alpha}{t}x\eta(x) = -\eta'(x) + c_1,$$

which is a linear first order equation. Write it as

$$\eta'(x) + \frac{\alpha}{t}x\eta(x) = c_1$$

and multiply through by the integrating factor $e^{\frac{\alpha x^2}{2t}}$. We now have

$$\frac{d}{dx}(e^{\frac{\alpha x^2}{2t}}\eta(x)) = c_1 e^{\frac{\alpha x^2}{2t}}$$

and so we can integrate in x , add a constant of integration c_2 , and do some algebra to find that

$$\eta(x) = c_1 e^{-\frac{\alpha x^2}{2t}} \int e^{\frac{\alpha x^2}{2t}} dx + c_2 e^{-\frac{\alpha x^2}{2t}}.$$

Let's now go back to the original ϕ notation (recall $\eta(x) = \phi(x/t^\alpha)$) so that

$$\phi\left(\frac{x}{t^\alpha}\right) = c_1 e^{-\frac{\alpha x^2}{2t}} \int e^{\frac{\alpha x^2}{2t}} dx + c_2 e^{-\frac{\alpha x^2}{2t}}.$$

It should now be apparent what α has to be: the right side is a function of x^2/t , or equivalently, x/\sqrt{t} , so if this is going to work we need $\alpha = 1/2$. Also, we want ϕ to die out as x gets large. The term with the c_1 coefficient looks like trouble. You can convince yourself that it in fact blows up for large x . Let's set $c_1 = 0$ (and remember, $\alpha = 1/2$) to obtain

$$\phi\left(\frac{x}{\sqrt{t}}\right) = c_2 e^{-\frac{x^2}{4t}}.$$

It's easy to see that this function dies out rapidly for large x . This gives us a prospective solution to the heat equation

$$\psi(x, t) = c_2 \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}}$$

for any constant c_2 . It works! You can plug ψ into the heat equation and check this. The last thing we need to do is adjust c_2 to obtain $\int_{-\infty}^{\infty} \psi(x, t) dx = 1$. It's a famous fact that

$$\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}.$$

Armed with this fact, you can do a simple change of variable $u = x/\sqrt{t}$ to find that

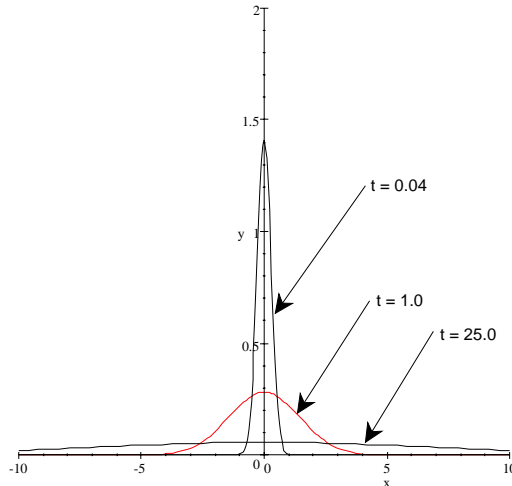
$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}} dx = 2\sqrt{\pi}$$

so we should choose $c_2 = \frac{1}{2\sqrt{\pi}}$. Thus we have

$$\psi(x, t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}. \quad (16)$$

This function $\psi(x, t)$ is of the form $\frac{1}{\sqrt{t}}\phi(x/\sqrt{t})$ with $\phi(u) = \frac{1}{2\sqrt{\pi}}e^{-u^2/4}$.

The function $\psi(x, t)$ defined by equation (16) is called the *Green's Function*, or *Green's kernel*, or *fundamental solution* for the heat equation. It is indeed a mollifier as described above. Here are some plots of $\psi(x, t)$ as a function of x for various t . When $t = 0.04, 1$, and 25 we have



As t gets small we should get something like a “delta” function, which appears to be happening.

From the previous arguments, we now know that the function $u(x, t)$ defined by equation (12) with $\psi(x, t)$ defined by equation (16) satisfies the heat equation with the proper initial condition. This is exactly

$$u(x, t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} f(y) dy.$$

It’s worth noting that the Green’s function $\psi(x, t)$ is in some sense the solution to the heat equation with initial condition $\psi(x, 0) = \delta(x)$, where $\delta(x)$ is the usual Dirac delta function, an infinite spike at position zero. In the language of engineering $\psi(x, t)$ is the impulse response of this system. The figure above does a good job of illustrating the nature of $\psi(x, t)$. Consider $\psi(x, t)$ for a very small t —it’s very high, but thin—the heat from the delta function spike at $x = 0$ has just begun to spread out. As t increases the heat continues spreading out, and that’s exactly what you see in the figure.

Existence and Regularity of Initial Data

Equation (2) shows that a solution to the heat equation exists. What do we need from the initial data f in order for the integral defining $u(x, t)$ to make sense? Certainly if f is bounded and continuous, the integral is

perfectly well-defined; moreover, the function $e^{-\frac{x^2}{4t}}$ and all of its derivatives decay so very rapidly as $|x| \rightarrow \infty$, slipping $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t}$ operators under the integral won't be an issue (so u really will satisfy the heat equation). So we have our existence result:

Theorem 1 *The heat equation with bounded continuous initial data $f(x)$ has a solution for $-\infty < x < \infty$ and $t > 0$, defined by equation (2), and $u(x, t)$ satisfies $u(x, 0) = f(x)$ in the sense that*

$$\lim_{t \rightarrow 0^+} u(x, t) = f(x).$$

Actually, you can put in functions f which aren't continuous or bounded; all you really need is that f doesn't grow too fast (faster than e^{x^2}) or be so discontinuous that the integral doesn't make sense.

Problems

- Show that all x and t derivatives (mixed or not) of any order of the function $\psi(x, t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}$ are just sums of terms of the form

$$ct^{k+1/2} x^m \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}$$

for integers k and m and some constant c .

- Show that any derivative of $\psi(x, t)$ of any order in x and/or t approaches zero as $|x| \rightarrow \infty$.

0.4 Interesting Properties of the Solution

The heat equation has some interesting qualitative features that are quite different from, for example, the wave equation.

Speed of Propagation/Causality

Recall that for the wave equation information propagated at a finite speed. But look at equation (2)—suppose that the initial condition $f(x)$ is zero everywhere except on some small interval (a, b) . Because the exponential function is never zero, the integral typically won't be zero, even for small t and

x arbitrarily far from (a, b) . More generally, the initial condition f affect the solution $u(x, t)$ for all x , no matter how small t is. *The heat propagates with infinite speed.*

Smoothing Properties

First, recall that a function is said to be C^∞ if it is infinitely differentiable—you can differentiate as many times as you like and the derivatives never develop corners or discontinuities. The function $\frac{1}{2\sqrt{\pi t}}e^{-\frac{(x-y)^2}{4t}}$ is such a function, with respect to all variables. Also, for any $t > 0$ this function decays to zero VERY rapidly away from $x = y$ (see the last problem). As a result, we can differentiate both sides of equation (2) with respect to x as many times as we like, by slipping the x derivative inside the integral to find

$$\frac{\partial^n u}{\partial x^n}(x, t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \frac{\partial^n (e^{-\frac{(x-y)^2}{4t}})}{\partial x^n} f(y) dy.$$

The integral always converges (if, e.g., f is bounded). In short, no matter what the nature of f (discontinuous, for example) the solution $u(x, t)$ will be C^∞ in x , for any time $t > 0$; the initial data is “instantly” smoothed out! Contrast this to the wave equation—imagine a square wave moving down the x axis—the discontinuity will never dissipate, it just propagates along at some fixed speed.

0.5 Uniqueness and Stability

The L^2 Norm

For the wave equation we measured how close two functions f and g are on some interval $a < x < b$ (either a or b can be $\pm\infty$) using the supremum norm $\|f - g\|_\infty = \sup_{a < x < b} |f(x) - g(x)|$ (we can also talk about the size or “norm” of a single function, as $\|f\|_\infty = \sup_{a < x < b} |f(x)|$). There’s another common way to measure the distance between two functions on an interval (a, b) , namely

$$\|f - g\|_2 = \left(\int_a^b (f(x) - g(x))^2 dx \right)^{1/2}.$$

We can also measure the norm of a function with

$$\|f\|_2 = \left(\int_a^b (f(x))^2 dx \right)^{1/2}.$$

The latter quantity is called the “ L^2 norm”, so the distance between two functions can be measured as the L^2 norm of their difference. It’s easy to see that if $\|f\|_2 = 0$ then f must be the zero function, and $\|f - g\|_2 = 0$ implies $f = g$.

The L^2 norm is very much like the usual Pythagorean norm for vectors from linear algebra (or just Calc I), which for a vector $\mathbf{x} = \langle x_1, \dots, x_n \rangle$ is defined as

$$\|\mathbf{x}\|_2 = \left(\sum_{j=1}^n x_j^2 \right)^{1/2}.$$

The distance between two vectors or points \mathbf{x} and \mathbf{y} in space is then just $\|\mathbf{x} - \mathbf{y}\|_2$. It turns out that the L^2 norm is sometimes more suitable for measuring the distance between two functions. It’s what we’ll use to derive stability results (and uniqueness) for the heat equation.

Uniqueness and Stability

If $u(x, t)$ satisfies the heat equation, define the quantity

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty} u^2(x, t) dx,$$

(assuming the integral is finite) similar to the energy for the wave equation. It’s not clear if $E(t)$ above is physically any kind of “energy”, but I don’t care; it can be used to prove uniqueness and stability for the heat equation.

It’s easy to compute

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} u(x, t) u_t(x, t) dx,$$

provided u and u_t decay fast enough at infinity (and are, say, continuous) which we’ll assume. Replace u_t in the integral with u_{xx} (since $u_t - u_{xx} = 0$) to find

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} u(x, t) u_{xx}(x, t) dx.$$

Now integrate by parts (take an x derivative off of u_{xx} , put it on u) to find

$$\frac{dE}{dt} = \lim_{x \rightarrow \infty} u_x(x, t)u(x, t) - \lim_{x \rightarrow -\infty} u_x(x, t)u(x, t) - \int_{-\infty}^{\infty} u_x^2(x, t) dx.$$

We'll assume that u limits to zero as $|x| \rightarrow \infty$ —in fact, we'll actually impose it as a requirement! Assuming u_x stays bounded, we can drop the endpoint limits in dE/dt above (alternatively, we could assume u_x limits to zero and u stays bounded). We obtain

$$\frac{dE}{dt} = - \int_{-\infty}^{\infty} u_x^2(x, t) dx. \quad (17)$$

It's obvious that $dE/dt \leq 0$, so $E(t)$ is a non-increasing (and probably decreasing) function of t . Thus, for example, $E(t) \leq E(0)$ for $t > 0$.

Of course $E(t)$ is simply the L^2 norm of the function $u(x, t)$, as a function of x at some fixed time t , and we've shown that this quantity cannot increase in size over time. Here's a bit of notation: we use $\|u(\cdot, t)\|_2$ to mean the L^2 norm of u as a function of x for a fixed value of t . With this notation we can formally state the following Lemma:

Lemma 1 *Suppose that $u(x, t)$ satisfies the heat equation for $-\infty < x < \infty$ and $t > 0$ with initial data $u(x, 0) = f(x)$. Then*

$$\|u(\cdot, t)\|_2 \leq \|f\|_2.$$

We can wipe out uniqueness and stability at the same time. Consider two solutions u_1 and u_2 to the heat equation, with initial data f_1 and f_2 . Form the function $u = u_1 - u_2$, which has initial data $f_1 - f_2$. Apply Lemma 1 to u and you immediately obtain

Theorem 2 *If u_1 and u_2 are solutions to the heat equation for $-\infty < x < \infty$ and $t > 0$ with initial data $u_1(x, 0) = f_1(x)$ and $u_2(x, 0) = f_2(x)$, and both u_1 and u_2 decay to zero in x for all $t > 0$, then*

$$\|u_1(\cdot, t) - u_2(\cdot, t)\|_2 \leq \|f_1 - f_2\|_2.$$

To prove uniqueness take $f_1 = f_2$ to conclude that at all times $t > 0$ we have $\|u_1(\cdot, t) - u_2(\cdot, t)\|_2 = 0$, or

$$\int_{-\infty}^{\infty} (u_1(x, t) - u_2(x, t))^2 dx = 0.$$

This forces $u_1 - u_2 = 0$, so $u_1(x, t) = u_2(x, t)$ for all x and t .