More on the Heat Equation

MA 436
Kurt Bryan

Recap

We've solved the heat equation $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$ on the interval $0 < x < 1$ with initial condition $u(x, 0) = f(x)$ for $f \in L^2(0, 1)$ and boundary conditions $u(0, t) = u(1, t) = 0$. The same procedure allows us to solve on a general interval $a < x < b$, though it's just a bit messier.

Today we'll look at a few variations on the heat equation, boundary and initial conditions.

Zero Neumann Boundary Conditions

Consider solving the heat equation for $0 < x < 1$ with $u(x, 0) = f(x)$ for $f \in L^2(0, 1)$ and Neumann boundary conditions $u_x(0, t) = u_x(1, t) = 0$. Recall that separable solutions to the heat equation must be of the form $ce^{-\lambda^2 t}\cos(\lambda x)$ or $ce^{-\lambda^2 t}\sin(\lambda x)$ for some λ. The Neumann boundary condition at $x = 0$ this time eliminates the choice $ce^{-\lambda^2 t}\sin(\lambda x)$. We're left with $ce^{-\lambda^2 t}\cos(\lambda x)$, and the condition $u_x(1, t) = 0$ forces $\lambda = k\pi$, just like before. Since $\cos(-x) = \cos(x)$, it's not hard to see we need only consider $k \geq 0$. We thus seek solutions of the form $u(x, t) = \sum_k c_k e^{-k^2\pi^2 t}\cos(k\pi x)$. The family $\phi_k(x) = \sqrt{2}\cos(k\pi x)$ for $k \geq 1$, with $\phi_0(x) = 1$, forms a complete orthonormal family in $L^2(0, 1)$. We can thus construct solutions as

$$u(x, t) = \sum_{k=0}^{\infty} c_k e^{-k^2\pi^2 t}\phi_k(x).$$

Choose the c_k as $c_k = (f, \phi_k)$ for $k \geq 0$. The same analysis as before shows that this really does define a solution to the heat equation with the proper boundary and initial conditions.

The Non-Homogeneous Heat Equation

Consider trying to solve the non-homogeneous heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = F(x, t) \qquad (1)$$

for $0 < x < 1$, $0 < t$, for some function F, with boundary conditions $u(0,t) = u(1,t) = 0$ and initial condition $u(x,0) = f(x)$.

First, we'll work with the orthonormal family $\phi_k(x) = \sqrt{2}\sin(k\pi x)$ on $L^2(0,1)$. For each fixed time t we can write out a Fourier series for $F(x,t)$, as a function of x, as

$$F(x,t) = \sum_{k=1}^{\infty} c_k(t)\phi_k(x) \tag{2}$$

where $c_k(t) = (F(\cdot,t),\phi_k)$, where $(F(\cdot,t),\phi_k)$ means the inner product of $F(x,t)$ (as a function of x) with ϕ_k. Note that c_k really does depend on t. It seems clear that we should require the function $F(x,t)$ to be in $L^2(0,1)$, as a function of x, for each fixed $t > 0$.

We seek a solution $u(x,t)$ to the non-homogeneous heat equation as

$$u(x,t) = \sum_{k=1}^{\infty} q_k(t)\phi_k(x) \tag{3}$$

for appropriately chosen functions $q_k(t)$. In fact, if we apply the heat operator to $u(x,t)$ (term by term, and use $\phi_k'' = -k^2\pi^2\phi_k$) we obtain

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \sum_{k=1}^{\infty}(q_k'(t) + k^2\pi^2 q_k(t))\phi_k(x). \tag{4}$$

We want $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = F$, so match the right sides of equations (2) and (4); specifically, match each ϕ_k coefficient to obtain

$$q_k'(t) + k^2\pi^2 q_k(t) = c_k(t) \tag{5}$$

for $k \geq 0$, an infinite family of linear, constant coefficient, first order ODE's for unknown functions $q_k(t)$. Each such ODE requires an initial condition, which comes from $u(x,0) = f(x)$. Specifically, plug $t = 0$ into (3) to find we need

$$\sum_{k=1}^{\infty} q_k(0)\phi_k(x) = f(x)$$

so we should choose $q_k(0) = (f,\phi_k)$ for $k \geq 0$.

So the prescription for solving equation (1) with boundary conditions $u(0,t) = u(1,t) = 0$ and initial condition $u(x,0) = f(x)$ is this: Compute the functions $c_k(t) = (F(\cdot,t),\phi_k)$, then solve each of the linear ODE's (5) with initial condition $q_k(0) = (f,\phi_k)$. The solution $u(x,t)$ is given by equation (3).

By the way, it's easy to solve equation (5); just use integrating factor $e^{k^2\pi^2 t}$.

General Boundary Conditions

Let's look at solving $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = F$ with $u(x,0) = f(x)$ and general Dirichlet boundary conditions $u(0,t) = h_0(t), u(1,t) = h_1(t)$, where h_0 and h_1 are some given functions. This is really easy, given the computations we've already done.

Let $v(x,t)$ be any function which satisfies the boundary conditions $v(0,t) = h_0(t), v(1,t) = h_1(t)$, say

$$v(x,t) = h_0(t) + x(h_1(t) - h_0(t)).$$

Note that $\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = h_0'(t) + x(h_1'(t) - h_0'(t))$; define $G(x,t) = h_0'(t) + x(h_1'(t) - h_0'(t))$. Also let $g(x) = v(x,0) = h_0(0) + x(h_1(0) - h_0(0))$ (so g is the initial condition for v). Let $w(x,t)$ be the solution to $\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = F - G$ with initial condition $w(x,0) = f(x) - g(x)$ and boundary conditions $w(0,t) = w(1,t) = 0$. Note that we know how to construct w from the computations of the last section.

You can easily check that the solution u we seek is given by $u = v + w$. Of course all of this requires that the functions F, f, h_0, and h_1 satisfy some modest regularity conditions, but we won't dwell on that.

A very similar procedure can be used to obtain general Neumann boundary conditions.