The Dirac Delta Function

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Impulsive Inputs and Impulse Response

Consider a spring-mass system with a time-dependent force $f(t)$ applied to the mass. The situation is modelled by the second-order differential equation

$$mx''(t) + cx'(t) + kx(t) = f(t) \tag{1}$$

where t is time and $x(t)$ is the displacement of the mass from equilibrium. Now suppose that for $t \leq 0$ the mass is at rest in its equilibrium position, so $x'(0) = x(0) = 0$. At $t = 0$ the mass is struck by an "instantaneous" hammer blow. This situation actually occurs frequently in practice—a system sustains an forceful, almost-instantaneous input. Our goal is to model the situation mathematically and determine how the system will respond.

In the above situation we might describe $f(t)$ as a large constant force applied on a very small time interval. Such a model leads to the forcing function

$$f(t) = \begin{cases} \frac{A}{\epsilon}, & 0 \leq t \leq \epsilon \\ 0, & \text{else} \end{cases} \tag{2}$$

Here A is some constant and ϵ is a "small" positive real number. When ϵ is close to zero the applied force is very large during the time interval $0 \leq t \leq \epsilon$ and zero afterwards.

In this case it's easy to see that for any choice of ϵ we have

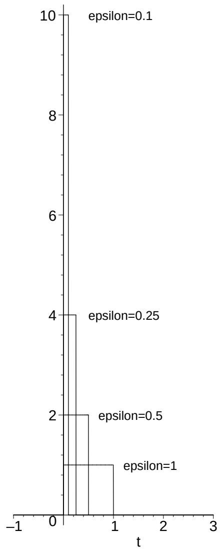
$$\int_{-\infty}^{\infty} f(t)\, dt = A,$$

a quantity which is called the total *impulse* delivered by the hammer blow, with units of force times time. In what follows let's normalize the impulse by taking $A = 1$. With this normalization we can write f in terms of Heaviside functions, as

$$f_\epsilon(t) = \frac{1}{\epsilon}(H(t) - H(t-\epsilon)) \tag{3}$$

where I've now subscripted f with ϵ to explicitly denote that dependence. Our ultimate interest is the behavior of the solution to equation (1) with forcing function f_ϵ in the limit that $\epsilon \to 0$.

However, it's easy to see that in the limit that $\epsilon \to 0$, $f_\epsilon(t)$ makes no sense as a function. You can easily check that for any $t \neq 0$, $\lim_{\epsilon\to 0} f_\epsilon(t) = 0$, while $\lim_{\epsilon \to 0} f_\epsilon(0)$ is undefined. The appearance of $f_\epsilon(t)$ for various ϵ is shown below.



We'll have to make sense of $\lim_{\epsilon \to 0} f_\epsilon$ in some other way.

Let $x_\epsilon(t)$ denote the solution to equation (1) with $f(t) = f_\epsilon(t)$. For $\epsilon > 0$ we can solve the resulting initial value problem using Laplace transforms. To do this we need to compute the Laplace transform of $f_\epsilon(t)$, given by the integral

$$\mathcal{L}\{f_\epsilon\} = \int_0^\infty e^{-st} f_\epsilon(t)\, dt = \frac{1}{\epsilon} \int_0^\epsilon e^{-st}\, dt = \frac{1 - e^{-\epsilon s}}{\epsilon s} \tag{4}$$

Now note that even though $\lim_{\epsilon \to 0} f_\epsilon$ make no sense as a function, the Laplace transform of $\mathcal{L}\{f_\epsilon\}$ is perfectly sensible as ϵ approaches zero, and in fact we find that

$$\lim_{\epsilon \to 0} \mathcal{L}\{f_\epsilon\}(s) = \lim_{\epsilon \to 0} \frac{1 - e^{-\epsilon s}}{\epsilon s} = 1 \tag{5}$$

for any $s > 0$.

Exercise

- Prove equation (5). (Hint: L'Hopital's rule).

Applying the Laplace Transform to both sides of (1) and using the zero initial conditions results in

$$ms^2 X_\epsilon(s) + csX_\epsilon(s) + kX_\epsilon(s) = \frac{1 - e^{s\epsilon}}{s\epsilon}$$

where $X_\epsilon = \mathcal{L}\{x_\epsilon\}$. We find that

$$X_\epsilon(s) = \frac{1}{ms^2 + cs + ks} \frac{1 - e^{s\epsilon}}{s\epsilon} \tag{6}$$

and by letting $\epsilon \to 0$ and using equation (5) we have

$$X(s) = \frac{1}{ms^2 + cs + ks} \tag{7}$$

where $X(s) = \lim_{\epsilon \to 0} X_\epsilon(s)$. Now inverse transform $X(s)$ to find a function $x(t)$, our solution to equation (1) in which the forcing function is an instantaneous, "infinitely" forceful input which delivers a total impulse of one to the system. The resulting solution $x(t)$ is called the *impulse response* of the system.

Exercises

- Take $m = 1$, $c = 3$, and $k = 2$. Solve equation (1) with zero initial conditions and forcing function f_ϵ for $\epsilon = 1, 0.1, 0.01$, and also for "$\epsilon = 0$" by inverting the transform in equation (7). Compare the solutions $x_\epsilon(t)$ with the solution $x(t)$ obtained from equation (7).

- Suppose that instead of computing $X(s) = \lim_{\epsilon \to 0} X_\epsilon(s)$ and then inverting X to obtain $x(t)$, we instead first invert X_ϵ to obtain $x_\epsilon(t)$ and THEN take the limit in ϵ. Show this gives the same result, that is, $x(t) = \lim_{\epsilon \to 0} x_\epsilon(t)$.

The Delta Function

The entire process we went through above to find the impulse response of the system is often abbreviated, a bit non-rigorously, in the following way. The limit $\lim_{\epsilon \to 0} f_\epsilon(t)$ is meaningless as a function, but people nonetheless often write

$$\lim_{\epsilon \to 0} f_\epsilon(t) = \delta(t). \tag{8}$$

The object $\delta(t)$ on the right above is called the *Dirac Delta Function*, or just a *delta function* for short. Conceptually, it represents a function which is zero for all t except $t = 0$, where it's "infinite" in just the right way that it represents a unit impulse. Since the total impulse delivered to the system is just the integral of the input over time, this means that

$$\int_a^b \delta(t)\, dt = 1 \tag{9}$$

where (a, b) is ANY interval which contains 0.

Now defining a function in this way might look like nonsense, and it is, but it can be put on a more rigorous footing. If you want to see that done (and see a lot more useful applications of delta functions), take a PDE course, e.g., MA 302. We'll take a fairly casual, intuitive approach. One rule to keep in mind is that delta functions should generally appear under integrals, or at least they should appear under an integral at some point in the computations, typically something like

$$\int_a^b \delta(t) g(t)\, dt$$

for some continuous function $g(t)$ on some interval (a, b). The above expression is to be interpreted rigorously as

$$\int_a^b \delta(t) g(t)\, dt = \lim_{\epsilon \to 0} \int_a^b f_\epsilon(t) g(t)\, dt. \tag{10}$$

In fact, suppose that $a < 0 < b$. Then for $\epsilon < b$ the right side of equation (10) yields

$$\int_a^b \delta(t)g(t)\,dt = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^\epsilon g(t)\,dt.$$

Now imagine evaluating the integral on the right: You'd find an antiderivative G for g, that is, a function G such that $G' = g$. In this case the integral on the right is just $G(\epsilon) - G(0)$ and we have

$$\int_a^b \delta(t)g(t)\,dt = \lim_{\epsilon\to 0}\frac{G(\epsilon) - G(0)}{\epsilon} = G'(0) = g(0). \tag{11}$$

where we've simply used the definition of the derivative $G'(0)$ (and the assumption that g is continuous at $t = 0$).

Exercises

- Apply the same reasoning to evaluate $\int_a^b \delta(t)g(t)\,dt$ if the interval $[a, b]$ does not contain 0.
- What is $\int_0^\infty \delta(t-c)g(t)\,dt$?

The Laplace Transform and Delta Functions

In light of equation (11), the Laplace transform of $\delta(t)$ should be given by

$$\int_0^\infty \delta(t)e^{-st}\,dt = e^0 = 1$$

in perfect accordance with equation (5).

The entire solution procedure with delta function input is illustrated by the following example:

Example: A spring-mass system with mass 2, damping 4, and spring constant 10 is subject to a hammer blow at time $t = 0$. The blow imparts a total impulse of 1 to the system, which as initially at rest. Find the response of the system.

The situation is modelled by

$$2x''(t) + 4x'(t) + 10x(t) = \delta(t)$$

with $x(0) = 0$ and $x'(0) = 0$. Laplace transform both sides and use the initial conditions to obtain

$$2s^2X(s) + 4sX(s) + 10X(s) = 1$$

from which we find that $X(s) = \frac{1}{2s^2+4s+10}$. You can inverse transform $X(s)$ to find that the response is

$$x(t) = \frac{1}{4}e^{-t}\sin(2t).$$

Exercise

- A salt tank contains 100 liters of pure water at time $t = 0$, when salty water begins flowing into the tank at 2 liters per second. The incoming liquid contains 1/2 kg of salt per liter. The well-stirred liquid flows out of the tank at 2 liters per second.
  - Model the situation with a first order DE, and find the amount of salt in the tank at any time.
  - Suppose that at time $t = 20$ seconds 5 kg of salt is instantaneously dumped into the tank. Modify the DE from the previous part (hint: a delta function appears on the right side). Solve the resulting DE using the Laplace transform. Plot the solution to make sure it's sensible.

Some Delta Function Calculus

Despite it's strange definition, the delta function can be integrated and differentiated, although not the traditional sense—rather, there is a useful interpretation of what it means to integrate and differentiate a delta function.

Here's how to interpret the anti-derivative of $\delta(t)$. Let $H(t)$ be the Heaviside function. It's easy to see that for $a > 0$ and any continuous function g we have

$$\int_{-\infty}^{a} H(t)g(t)\,dt = \int_{0}^{a} g(t)\,dt. \tag{12}$$

Let G be an anti-derivative for g. We can compute the left side of equation (12) using integration by parts ($\int u\,dv = uv = \int v\,du$) with $u = H(t), dv = g(t)\,dt$, ignoring the fact that H is not differentiable at $t = 0$. After the integration by parts we find the left side of (12) is given by

$$G(a) - \int_{-\infty}^{a} G(t)H'(t)\,dt.$$

The right side of (12) is just $G(a) - G(0)$. Equating this with the last displayed expression produces

$$\int_{-\infty}^{a} G(t)H'(t)\,dt = G(0).$$

But this is just equation (11) (with g replaced by G). In short, $H'(t)$ under an integral acts just like $\delta(t)$, and hence we interpret $H'(t) = \delta(t)$. The Heaviside function is the anti-derivative of the delta function.

Now consider differentiating a delta function. The usual definition of a derivative

$$\lim_{h\to 0} \frac{\delta(t+h) - \delta(h)}{h}$$

makes no sense here, at least not when $t = 0$. Instead, we'll define δ' as above, via integration by parts. Contemplate the integral

$$\int_{-\infty}^{\infty} \delta'(t)g(t)\,dt$$

for some continuously differentiable function g. Integrating by parts with $u = g$, $dv = \delta'(t)\,dt$ gives

$$\int_{-\infty}^{\infty} \delta'(t)g(t)\,dt = -\int_{-\infty}^{\infty} \delta(t)g'(t)\,dt = -g'(0). \tag{13}$$

This last equation is the defining property of $\delta'(t)$.

Exercise

- Another way to make sense of $\delta'(t)$ is as

$$\int_{-\infty}^{\infty} \delta'(t)g(t)\,dt = \lim_{h\to 0}\int_{-\infty}^{\infty} \frac{\delta(t+h)-\delta(h)}{h} g(t)\,dt$$

  for some continuously differentiable function g. Show it gives the same result as equation (13).

Convolution with Delta Functions

Consider convolving a delta function $\delta(t)$ with a continuous function $g(t)$. Using the basic properties of the delta function, we find

$$(\delta * g)(t) = \int_0^t \delta(\tau)g(t-\tau)\,d\tau = g(t)$$

so that $\delta * g = g$! The delta function is the "identity" with respect to convolution. This also makes perfect sense with regard to the Laplace transform, for $\mathcal{L}\{\delta * g\} = \mathcal{L}\{\delta\}\mathcal{L}\{g\} = \mathcal{L}\{g\}$.

More on Impulse Responses

Since $f * \delta = f$, we might write equation (1) as

$$mx''(t) + cx'(t) + kx(t) = f * \delta.$$

We'll still take zero initial conditions. Laplace transform both sides, using $F = \mathcal{L}\{f\}$, to find $ms^2X(s) + csX(s) + kX(s) = F(s)$ or

$$X(s) = \frac{F(s)}{ms^2 + cs + k} \tag{14}$$

Now let $x_0(t) = \mathcal{L}^{-1}\{1/(ms^2+cs+k)\}$ denote the impulse response of the system. From the basic properties of the inverse Laplace transform we find from equation (14) that $x = f * x_0$. Thus the time domain response of the system to an input forcing function $f(t)$ (with zero initial conditions) is simply the convolution of f with the impulse response function of the system.

Example: For a spring-mass system with $m = 2$, $c = 4$ and $k = 10$ we computed in an above example that the impulse response of the system was given by $x_0(t) = \frac{1}{4}e^{-t}\sin(2t)$. How will the system respond if we use a forcing function $f(t) = e^{-t}$ with zero initial conditions? From the above discussion the response of the system is given by

$$(f * x_0)(t) = \int_0^t \frac{1}{4}e^{-\tau}\sin(2\tau)e^{t-\tau}\,d\tau = \frac{1}{4}e^{-t}(\cos(t)-1)^2.$$

Parameter Identification

One of the most useful applications of delta functions and impulse responses is for *system* or *parameter* identification. Suppose you have some physical system which is modelled by an ODE, but the model contains certain unknown physical parameters. For example, you may have a spring-mass system governed by equation (1), but with unknown m, c, and k. In order to determine these parameters, you apply an impulse input $\delta(t)$ to the mass and observe the system's response. From this you try to infer the unknown physical parameters.

Suppose, for example that a unit impulse is applied to the mass in a spring-mass system, initially at equilibrium, and you observe the system respond as $x(t) = e^{-2t}\sin(3t)$. What are the physical parameters m, c, and k? Start with the model (1) with $f(t) = \delta(t)$ and Laplace transform both sides of the equation, then solve to find $X(s) = \frac{1}{ms^2+cs+k}$. But we know $x(t) = e^{-2t}\cos(3t)$ has Laplace transform $\frac{3}{s^2+4s+13} = \frac{1}{\frac{1}{3}s^2+\frac{4}{3}s+\frac{13}{3}}$, and so we conclude that $m = \frac{1}{3}$, $c = \frac{4}{3}$, and $k = \frac{13}{3}$.

Exercises

- A spring-mass system initially at equilibrium is subject to a delta function input at time $t = 0$, total impulse 1. The observed response of the system is given by $x(t) = \frac{1}{4}(e^{-3t} - e^{-5t})$. Determine m, c, and k.

- A spring-mass system with UNKNOWN initial conditions is subject to a delta function input at time $t = 0$, total impulse 1. The observed response of the system is given by $x(t) = 2e^{-2t} - e^{-4t}$. Is it possible to determine m, c, and k? How about the initial conditions? Recover whatever can be recovered!