Dimensional Analysis

Mathematical Modelling Week 2

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How does the escape velocity from a planet's surface depend on the planet's mass and radius? This sounds like a physics problem, but you can figure out a plausible answer without doing any real physics. The idea is to use *dimensional analysis*, one of the most basic and useful techniques in mathematical modelling.

Let's take v for the escape velocity, m for the mass of the planet, R for the planet radius, and, since it's probably relevant, let's toss in the universal gravitational constant G. I haven't specified any units yet, and I won't. The escape velocity v has the physical dimension *length* per *time*, which I'll denote by writing $[v] = LT^{-1}$, where L means length and T means time. Similarly, $[m] = M$, where M is mass. Then $[R] = L$ and you can check that $[G] = L^3T^{-2}M^{-1}$, by either looking it up in a CRC handbook or using the universal gravitational law $F = Gm_1m_2/r^2$. Note that the physical dimension of a quantity like mass is not the same as the system of units used to measure that physical quantity. For example, slugs, grams and kilograms are all different systems of units for measuring mass, but mass is a physical quantity independent of any system of units. A quantity is *dimensionless* if it's a pure number, without units. For example, π and 2 are dimensionless. So are slopes (rise over run is dimensionless!) and angles (the definition of "radian" involves the ratio of two lengths). The value of dimensionless quantities doesn't depend on the system of physical units.

Let's hypothesize that the escape velocity can be expressed as a formula of the form

$$v = km^\alpha R^\beta G^\gamma \tag{1}$$

where α, β and γ are numbers to be determined. The number k is some dimensionless constant. Dimensional analysis proceeds on the observation that physical laws have to balance dimensionally. Roughly speaking, no number of apples can equal any number of oranges. Thus in equation (1), whatever α, β and γ are, both sides of the equation must have the same dimension. The dimension of the left side is LT^{-1}; the dimension of the right side is

$$[km^\alpha R^\beta G^\gamma] = M^\alpha L^\beta L^{3\gamma} T^{-2\gamma} M^{-\gamma} = M^{\alpha-\gamma} L^{\beta+3\gamma} T^{-2\gamma}.$$

If the two sides are to balance dimensionally, we must have

$$\begin{aligned} \alpha - \gamma &= 0, \\ \beta + 3\gamma &= 1, \\ -2\gamma &= -1. \end{aligned}$$

The only solution is $\alpha = 1/2$, $\beta = -1/2$, $\gamma = 1/2$, and so we conclude that if a relation of the form (1) holds then

$$v = k\sqrt{\frac{Gm}{R}}, \tag{2}$$

where k is some dimensionless constant.

Exercise: Find a plausible formula for the lift (a force) generated by an airplane wing, in terms of the density of the surrounding air, the area of the wing, and the speed at which it moves through the air.

A Generalization

Now we'll make dimensional analysis a bit more systematic, and examine a powerful and general technique in this subject.

Let some physical situation be described by quantities $q_1, q_2, \ldots, q_n$. For example, in the escape velocity problem we might take $q_1 = v, q_2 = R, q_3 = m, q_4 = G$. We seek a physical law that relates the q_i, in the form

$$f(q_1, q_2, \ldots, q_n) = 0$$

for a suitable choice of the function f. For example, equation (2) can be written in this form with $f(v, G, m, R) = v - \sqrt{Gm/R} = 0$. But there are always other possible choices for f also, for example, $f(v, G, m, R) = v^2 - Gm/R = 0$ is an entirely equivalent way to write the law, or even $f(v, G, m, R) = e^{v^2 - Gm/R} - 1 = 0$. It's not hard to see that a physical law always has infinitely many algebraically equivalent mathematical expressions. We want to find any one such expression, by determining the function f.

One requirement we will impose is that the physical law should be *unit free*. A physical law is said to be *unit free* if it holds regardless of the system

of units used. For example, the escape velocity law $v = \sqrt{(\frac{Gm}{R})}$ is unit free, for it holds in any system of units. Similarly, the law for distance fallen in the earth's gravitational field, $d = 1/2gt^2$, is unit free, while $d = 4.9\frac{meters}{sec^2}t^2$ is valid only in certain systems of units (e.g., MKS).

Here is a central fact from dimensional analysis:

Buckingham's Pi Theorem

Suppose we have a unit free physical law in the form

$$f(q_1, q_2, \ldots, q_n) = 0$$

where the q_k are dimensional variables, and that from $q_1, \ldots, q_n$ we can form m new independent dimensionless variables $\Pi_1, \ldots, \Pi_m$ ($m \leq n$). Then $f(q_1, \ldots, q_n) = 0$ is equivalent to a physical law of the form

$$h(\Pi_1, \ldots, \Pi_m) = 0$$

for some function h.

What does *independent* mean? Intuitively, to say that a variable q is independent from variables $q_1, \ldots q_k$ means that the value of q cannot be determined from $q_1, \ldots q_k$. For example, the variables vR and v^2R^2 are NOT independent; if you know either, you know the other. However, vR and vR^2 are independent. You can't tell what vR^2 is by knowing vR, or vice versa. More precisely, we'll say q is independent of $q_1, \ldots q_k$ if q cannot be written in the form $q = q_1^{\alpha_1} \cdots q_m^{\alpha_m}$ for any choice of the exponents α_i. A slightly more sophisticated definition is given below.

Let's redo the escape velocity example using the Pi theorem. Our dimensional variables are v, R, m and G. According to the theorem, we should first look for all independent dimensionless variables which can be formed from these variables. Let

$$\Pi = v^\alpha R^\beta m^\gamma G^\delta.$$

Since Π should be dimensionless we need

$$[v^\alpha R^\beta m^\gamma G^\delta] = L^{\alpha+\beta+3\delta} T^{-\alpha-2\delta} M^{\gamma-\delta} = 1,$$

or

$$\begin{aligned} \alpha + \beta + 3\delta &= 0, \\ -\alpha - 2\delta &= 0, \\ \gamma - \delta &= 0. \end{aligned}$$

A little algebra shows that there are infinitely many possibilities for α,β,γ, and δ (there are 4 unknowns but only 3 equations) but that all of the solutions are of the form

$$\beta = \frac{1}{2}\alpha, \quad \gamma = -\frac{1}{2}\alpha, \quad \delta = -\frac{1}{2}\alpha,$$

i.e., we can choose any α, but the other variables are then forced to satisfy these relations. As a result,

$$\Pi = v^\alpha R^{\alpha/2} m^{-\alpha/2} G^{-\alpha/2} = \left(v\sqrt{\frac{R}{mG}} \right)^\alpha .$$

Note that different choices for α don't lead to new independent variables for Π. If we choose $\alpha = 1$, we obtain some variable Π, while $\alpha = 2$ merely leads to Π^2, which is not independent. Choosing a general constant α merely gives Π^α. To fix a specific choice, let's take $\alpha = 1$, and so our dimensionless variable is $\Pi = v\sqrt{R/(mG)}$.

According to the Pi theorem, any physical law involving the parameters v,R,m, and G can be expressed as $h(\Pi) = 0$. This means that Π itself must be constant—it's just a root of $h(x) = 0$—so that $\Pi = k$ for some dimensionless constant k, which is exactly the statement of equation (2).

A More Sophisticated Example

Consider a small spherical ball being pushed by a force F through a viscous liquid. We want to find the terminal velocity of the ball. Let's suppose that the velocity v depends on the force F, the density ρ of the liquid, the viscosity μ of the liquid and the radius R of the ball. The dimensions of each variable are $[v] = LT^{-1}$, $[F] = MLT^{-2}$, $[\rho] = ML^{-3}$, $[\mu] = ML^{-1}T^{-1}$ (look viscosity up in a CRC if you want to know why—roughly, viscosity is the internal "friction" of the liquid), and $[R] = L$. Let's see how many independent dimensionless variables we can form. Let

$$\Pi = v^\alpha F^\beta \rho^\gamma \mu^\delta R^\eta .$$

Do the algebra to figure out conditions on the exponents so that the right hand side is dimensionless; you will obtain

$$\begin{aligned} \alpha + \beta - 3\gamma - \delta + \eta &= 0, \\ -\alpha - 2\beta - \delta &= 0, \\ \beta + \gamma + \delta &= 0. \end{aligned} \tag{3}$$

This is a system of 3 equation in 5 unknowns; we expect to be able to pick 2 of the variables arbitrarily and then solve for the other 3. Let's take α and β fixed and solve for γ, δ, and η, to obtain

$$\gamma = \alpha + \beta, \quad \delta = -\alpha - 2\beta, \quad \eta = \alpha.$$

You should convince yourself that we can obtain exactly two independent dimensionless variables, say Π_1 and Π_2 by taking $\alpha = 1$,$\beta = 0$ and $\alpha = 0$,$\beta = 1$, respectively. Furthermore, any other combination, e.g., $\alpha = 3$ and $\beta = 2$, could be obtained by combining these solutions.

Before proceeding, it's worth noting that we can give a more careful definition of what we mean by "independent" dimensionless variables. The linear system (3) can be cast in matrix form as $\mathbf{Mx} = \mathbf{0}$, where $\mathbf{x} = [\alpha, \beta, \gamma, \delta, \eta]^T$ (the T means transpose) and $\mathbf{M}$ is the matrix

$$\mathbf{M} = \begin{bmatrix} 1 & -3 & -1 & 1 & 1 \\ -2 & 0 & -1 & 0 & -1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

When we solve $\mathbf{Mx} = \mathbf{0}$, we're looking for vectors in the nullspace of $\mathbf{M}$, and each such vector corresponds to a dimensionless variable we can construct. By "independent dimensionless variables" we mean that the corresponding nullspace vectors should be independent. So the process really comes down to finding a basis for the nullspace of $\mathbf{M}$!

Now back to the example. For $\alpha = 1$, $\beta = 0$ we obtain $\gamma = 1$, $\delta = -1$, and $\eta = 1$, so

$$\Pi_1 = \frac{vR\rho}{\mu}$$

while $\alpha = 0$ and $\beta = 1$ gives $\gamma = 1$, $\delta = -2$, $\eta = 0$, and

$$\Pi_2 = \frac{F\rho}{\mu^2}.$$

The Pi Theorem says that the physical law relating these variables can be written in the form

$$h(\Pi_1, \Pi_2) = 0$$

for some function h. This kind of formula defines Π_1 implicitly as a function of Π_2 (or vice-versa), i.e.,

$$\Pi_1 = g(\Pi_2)$$

for some function g. If we return to our original variables this means that

$$\frac{vR\rho}{\mu} = g\left(\frac{F\rho}{\mu^2}\right)$$

or

$$v = \frac{\mu}{R\rho} g\left(\frac{F\rho}{\mu^2}\right) \tag{4}$$

for some (unfortunately unknown) function g. The dimensionless variable Π_1 is of great importance in fluid mechanics, and is called the *Reynold's* number.

Actually, we can deduce even more from equation (4). Intuition says that if $F = 0$ then $v = 0$, so we conclude that $g(0) = 0$. The tangent line approximation says that then $g(x) \approx kx$ if x is close to zero, where k is some constant ($g'(0)$, actually). In other words, if F is small (or, more precisely, if the quantity $\frac{F\rho}{\mu^2}$ is small, say because μ is large) then $g(x) \approx kx$ produces

$$v \approx \frac{kF}{R\mu}.$$

This is actually quite amazing—we've found a formula for the terminal velocity of a ball moving through a viscous fluid and we really don't know anything about fluid mechanics!

Here's one other fine point that's worth mentioning. In any physical formula involving transcendental functions, the argument to the transcendental function is always dimensionless. For example, we often encounter expressions like $\sin(\omega t)$ in physics. Note that t has units of time and ω has units of reciprocal time (ω is a frequency). Similarly, in expressions like e^{-kt}, if t is time then k has units of reciprocal time. One way to see that the variable x in functions like $\sin(x)$ or e^x must be dimensionless is to note that such functions are defined by power series, e.g., $e^x = 1 + x + \frac{1}{2}x^2 + \cdots$. If x was not dimensionless the various powers would be dimensionally incompatible

and couldn't be added.

Problems:

1. What does dimensional analysis have to say about the area of a circle as a function of its radius?

2. If a quantity x has dimensions $M^\alpha T^\beta L^\gamma$, what are the dimensions of dx/dt? What are the dimensions of $\int x\,dt$? What are the dimensions of $\int x\,dV$, where dV means we integrate over some region in three dimensional space?

3. Let V denote the volume between two concentric spheres of radii r_1 and r_2. Use the Buckingham Pi Theorem to find a formula for V in terms of r_1 and r_2. Then work out the answer from geometry. Reconcile the answers.

4. Find a formula for the radius of a black hole in term of the black hole's mass m, the gravitational constant G, and the speed of light c.

5. How is the period of a pendulum related to its length, the mass of the bob, gravitational acceleration AND (this is unusual) the initial angle from which it is released (the angle is in radians, dimensionless)?

6. How is the top speed of a car related to the engine's power, the density of the surrounding air, and the "area" of the front of the car?

7. Find a formula for the speed of sound in a gas as a function of its pressure and density.

8. Compute the frequency of a string's vibration in term of its linear density, the tension in the string, and the length of the string.

9. How is the thrust of a rocket motor related to the velocity of the exhaust gases and amount of mass expelled by the engine per unit time?

10. Recall that the electric potential difference between two points is the energy required to move a unit charge from one point to the other. If we use Q to denote the physical quantity charge, it follows that an electric potential V has dimensions $[V] = ML^2T^{-2}Q^{-1}$. An electric

current I has dimensions QT^{-1} (charge per unit time). Recall that inductance L in a circuit is defined by the equation $V = LdI/dt$, while capacitance C is defined by $Q = CV$. Find a plausible formula for the frequency of oscillation of an LC circuit.

11. Coulomb's Law says that $F = kq_1q_2/r^2$, where F is the force between two charges q_1 and q_2, r is the distance between the charges, and k is some constant. What are the dimensions of k? Use this information to say something about the capacitance (remember $Q = CV$) between two flat plates with areas A, separated by a distance d.

12. Two objects, masses m_1 and m_2 orbit each other at some distance d. Find a formula for the orbital period.

13. An object of mass m_1 is dropped from some great height h above a planet of mass m_2, radius R. Find a formula for the time t it takes the object to fall to the planet surface. Also, find a formula for the speed v of the object at the moment of impact.

14. In deep water it is an observed fact that the speed v at which waves propagate depend on their wavelength λ and the height h of the wave. It seems reasonable that the density ρ of water (or whatever the liquid is) and gravitational acceleration g should factor in. Find a physical law relating these variables, and solve for v in terms of the other variables. Under the assumption that h is small relative to λ, approximate v in terms of the other variables.

15. In the Theory of General Relativity, any mass (like the sun) bends light rays which pass by the mass. Find a formula for the angular deflection (this is dimensionless) of a light ray which passes at a distance r from the center of a mass m. The gravitational constant G and speed of light c are also relevant.

16. A pipe of length P and cross sectional area A is filled with a liquid with density ρ, viscosity μ. A force F is applied to the fluid at one end of the pipe; the fluid flows out the other end at a rate of q (mass per time). Find a formula for q in terms of the other variables. What if we consider A and P fixed and let F be very close to zero—what does the formula for q look like then?

17. How does the average speed of the molecules in a gas depend on the temperature of the gas? You might need to specify the molecular mass of the gas, and use the universal gas constant.

Proof of the Buckingham Pi Theorem

I won't give a general rigorous proof, but rather consider a specific (but representative) case. Consider again the terminal velocity of a ball falling through a liquid. We have physical variables v, F, ρ, R, and μ, with $[v] = LT^{-1}$, $[F] = MLT^{-2}$, $[\rho] = ML^{-3}$, $[\mu] = ML^{-1}T^{-1}$, and $[R] = L$. We believe that a unit free physical law of the form

$$f(v, F, \rho, R, \mu) = 0 \tag{5}$$

holds. We will show that in fact any such law can also be expressed as

$$h(\Pi_1, \Pi_2) = 0 \tag{6}$$

where $\Pi_1 = \frac{vR\rho}{\mu}$ and $\Pi_2 = \frac{F\rho}{\mu^2}$ are the dimensionless variables we constructed.

The first thing to note is that the original set of variables $S_1 = (v, F, \rho, R, \mu)$ is equivalent to the set $S_2 = (\Pi_1, \Pi_2, \rho, R, \mu)$ (I just replaced v and F with Π_1 and Π_2, but I could have replaced any other two). When I say S_1 and S_2 are "equivalent" I mean that any variable in one set can be constructed from the variables in the other set. Obviously anything in S_2 can be built from S_1. To see that the converse is true, just note that $v = \frac{\mu\Pi_1}{R\rho}$ and $F = \frac{\mu^2\Pi_2}{\rho}$.

The function f in equation (5) is a function $f(x_1, x_2, x_3, x_4, x_5)$ of five variables. Define a function $h(x_1, x_2, x_3, x_4, x_5)$ as

$$h(x_1, x_2, x_3, x_4, x_5) = f(\frac{x_1x_5}{x_3x_4}, \frac{x_2x_5^2}{x_3}, x_3, x_4, x_5).$$

Then it's easy to check that $f(v, F, \rho, R, \mu) = h(\Pi_1, \Pi_2, \rho, R, \mu)$, so that equation (5) and the equation $h(\Pi_1, \Pi_2, \rho, R, \mu) = 0$ are equivalent. We will now use the fact that the law $h(\Pi_1, \Pi_2, \rho, R, \mu) = 0$ must be unit free to show that h doesn't actually depend on its last three arguments.

Imagine re-scaling our time variable by some amount r (i.e., we were measuring time in certain units, now we'll use r times those units). This would not change the numerical values of the dimensionless variables Π_1

and Π_2, but it will change the value of any physical variable that depends on time. In particular, it changes the value of μ (but not ρ or R). We would generally expect that changing the value of μ would change the value of $h(\Pi_1, \Pi_2, \rho, R, \mu)$, but since the physical law is unit free, we always have $h = 0$. We conclude that h doesn't depend on the fifth argument. More precisely, if we rescale time by a factor of r then μ changes to $r\mu$, and the value of h changes to $h(\Pi_1, \Pi_2, \rho, R, r\mu)$. But since h is constant (the law is unit free)

$$\frac{\partial}{\partial r} h(\Pi_1, \Pi_2, \rho, R, r\mu) = r \frac{\partial h}{\partial x_5}(\Pi_1, \Pi_2, \rho, R, r\mu) = 0$$

for any r and μ. We conclude that $\frac{\partial h}{\partial x_5} = 0$, i.e., h doesn't depend on x_5.

Now imagine re-scaling length by some factor r and simultaneously re-scaling mass by $r^{1/3}$; this changes the numerical value of R, but not ρ (I delicately balanced the changes)! Again, since the law is unit free the value of h remains zero, and so by an argument similar to that above, h must not depend on the fourth argument. Finally, we're down to a law $h(\Pi_1, \Pi_2, \rho) = 0$. If we re-scale mass or length the value of ρ changes (but not the dimensionless variables); since h cannot change value, we conclude that h doesn't depend on the third argument. Thus the law is really given by equation (6).

Implications for Experimental Design

Intelligent use of dimensional analysis can greatly cut down on the amount of work needed to determine the relationship among the relevant physical variables in a given situation.

For example, suppose we want to determine the precise relationship between v, ρ, μ, F, and r in the "balling falling through a viscous liquid" setting (without the approximation that v is small or the liquid very viscous). We could conduct the experiment for a wide variety of different choices for each of the variables and then try fitting a function to all of it, but that would be a massive undertaking. Even if we experiment with just two different levels of each of ρ, μ, F, r, that's still $2^4 = 16$ experiments. And it's not clear that we can easily "custom order" fluids with any given density and viscosity anyway. But there is probably no need to take so much data—we know from the dimensional analysis that the relationship is given by equation (4). If we can figure out the function g, we know the entire relationship.

And we have a simple way to figure out the function g. All we need to do is collect data for a variety of different experimental configurations in which we vary the argument $F\rho/\mu^2$ of g. This can be accomplished by varying just the force F (say, by using balls of different density or size). In essence, we conduct an experiment in which we measure all five variables, methodically altering the value of $F\rho/\mu^2$, and so "sample" the function g at points of our choosing. This gives a rough picture of the function g over some range of its input variable, and so gives us the physical law over that range. It takes a lot less data.