Differentiating Under the Integral

MA 466
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When can you differentiate under an integral? Specifically, when is the manipulation

$$\frac{d}{dt}\left(\int_E f(x,t)\,dx\right) = \int_E f_t(x,t)\,dx$$

valid?

This is a nice application of Dominated Convergence. Let $f(x,t)$ be defined for $x \in E \subset \mathbb{R}^n$ and $a \leq t \leq b$. Assume that

- For each $t \in I = [a,b]$ $f(x,t)$, is measurable and integrable in x. This means that the function

  $$F(t) = \int_E f(x,t)\,dx$$

  is well-defined.

- $\frac{\partial f}{\partial t}(x,t)$ exists and is integrable over $x \in E$ for each $t \in I$.

- There exists some $\phi(x)$ such that $|\frac{\partial f}{\partial t}(x,t)| \leq \phi(x)$ for all $t \in I$ and $x \in E$.

Then $F'(t)$ exists and

$$F'(t) = \int_E f_t(x,t)\,dx.$$

Proof: Note that for any fixed t_0 and any $h_n \to 0$ we have

$$\frac{F(t_0+h_n)-F(t_0)}{h_n} = \int_E \frac{f(x,t_0+h_n)-f(x,t_0)}{h_n}\,dx. \tag{1}$$

By the mean value theorem we have

$$g_n(x,t_0) := \frac{f(x,t_0+h_n)-f(x,t_0)}{h_n} = f_t(x,t_0+r_n)$$

where $|r_n| \leq |h_n|$. Since f_t is dominated by ϕ we have $|g_n| \leq \phi$ too. Note also that $g_n(x,t_0)$ converges to $f_t(x,t_0)$.

Now equation (1) is just

$$\frac{F(t_0 + h_n) - F(t_0)}{h_n} = \int_E g_n(x, t_0)\, dx.$$

Take the limit of both sides over n and invoke dominated convergence on the right. We get that the limit on the left exists for any $h_n \to 0$, and that

$$F'(t_0) = \int_E f_t(x, t_0)\, dx.$$

By the way, when would we have $|\frac{\partial f}{\partial t}(x, t)| \le \phi(x)$? Well, certainly if f_t is bounded and $m(E) < \infty$, almost always the case in practice.