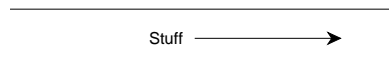# Conservation of Stuff

Mathematical Modelling Week 3

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The figure below represents some kind of conduit with "stuff" flowing through it. The conduit could be a pipe and the stuff could be water. The conduit could be a wire and the stuff could be electrons. The conduit could be a road and the stuff could be cars. A few of the possibilities are listed below.

Stuff ⟶

Conduit	Stuff
pipe	water
wire	current
road	cars
river	pollution
rod	heat

Our goal is to mathematically describe how stuff flows through the conduit. The description will vary a bit from situation to situation (cars and electrons are quite different) but each case has one important principle in common.

We'll consider the conduit to be one-dimensional and use the variable x as a coordinate to describe position along the conduit, with increasing x to the right. Since the situation in the conduit can change from moment to moment, we need to take into account time, denoted by t. Position x and time t are thus our independent variables.

Let's introduce two functions which describe the state of the stuff in the conduit. The first is the density function $\rho(x,t)$. The value of this function at the point x and time t is the linear density of the stuff in the conduit at that point and time. For example, if the conduit is a road, the density is the number of cars per unit length of road. If the conduit is a river and the stuff pollutants (probably dissolved in water) then the density is the amount of the pollutants (maybe measured by mass) in the water, per unit length of the river. The function $\rho(x,t)$ measures how tightly packed the stuff is and in general has units of "stuff" per unit length.

The second function we introduce is $q(x,t)$, the rate-of-flow function. This function describes how much stuff per second is flowing past the point x at time t from left to right. Its "units" are stuff per second. A positive value for q indicate stuff flowing from the left to the right. If the conduit is a road, q measures cars per second. If the conduit is a pipe, q measures liters per second.

Most of the equations we examine for the next few weeks will follow from one simple physical principle:
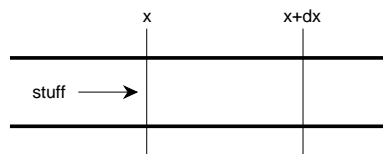
stuff is conserved

From this simple principle comes many of the mathematical equations that describe the world. Thus, if our conduit is a road, the conservation law says that cars are neither created nor destroyed, at least on the portion of road under examination. Similarly, water does not mysteriously vanish from our pipes. Stuff is conserved.

This is not to imply that we can't model leaky pipes or off-ramps. In such cases the conservation law can be extended to account for the loss or gain, but for now we'll stick to the case where no stuff is created or destroyed.

Thinking problem:

- You're a security guard at a high-tech, super secret office building. All exits and entrances have security cameras that let you monitor people going in and out. During the course of an hour you observe 18 people leave and 7 people enter. What is the net change in the number of people in the building?

- The security cameras are broken. However, each room of the building has sophisticated infrared sensors that let you determine exactly how many people are in each room. No one can escape detection. Now how would you determine the net change in the number of people in the building during that hour?

- Would you expect the number arrived at by the two previous procedures to be the same? Why? What assumptions have you made? What would you think if the numbers were different?

Time for some mathematics. Consider a small portion of the conduit, stretching from x to $x + \Delta x$ as pictured below. We'll call this the *control volume*.

Mental Exercise 1

- How much stuff is there in that small interval at a fixed time t? (Hint: use the density function, assume dx is small.)

Mental Exercise 2

- How fast is the amount of stuff in the interval *changing* at time t? Use the answer from Mental Exercise 1.

In Mental Exercise 2 you figured out how fast the amount of stuff in the control volume is change in terms of ρ. But there is another way to figure out how fast the amount of stuff in the control volume is changing, in terms of the function q.

Mental Exercise 3

- Visualize the flow as being from left to right (though it can go the other way too). At what rate is stuff flowing into the interval at x, in terms of q? At what rate is it flowing out at $x + dx$?

Mental Exercise 4

- How should the answers to Mental Exercises 2 and 3 be related? Write out a specific equation to express this fact. What assumptions did you just make?

Mental Exercise 5

- Given that the equation in Mental Exercise 4 must hold for all choices of dx, what relation must hold between ρ and q?

Homework

1. Think of a physical situation which can be modelled as "stuff" flowing through a conduit. What would $\rho(x,t)$ and $q(x,t)$ represent? What are their units?

2. Modify the derivation of the conservation equation above in Mental Exercise 5 to deal with the situation in which "stuff" is created or destroyed. Specifically, suppose $k(x,t)$ is a function which measures how fast stuff is created ($k > 0$) or destroyed ($k < 0$) at the point x at time t, in units of stuff per unit length of the conduit per unit time. You'll know you're on the right track if the units in the equation balance.