

Conservation Principles in Two Dimensions

Mathematical Modeling Week 9

Kurt Bryan

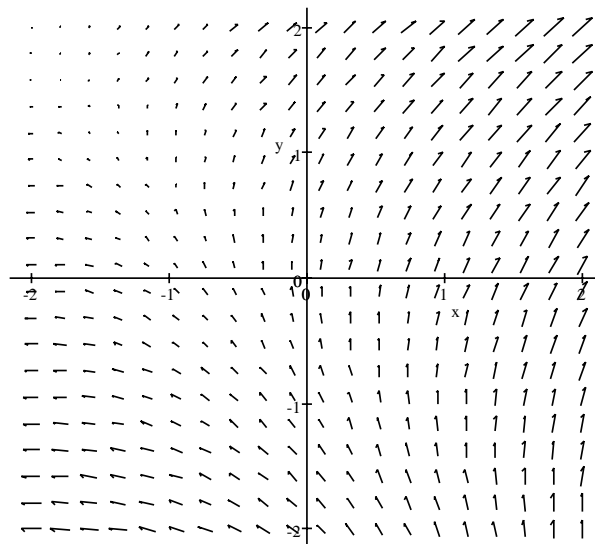
Conservation in Two Dimensions

Consider a two-dimensional region D in which some quantity flows. I'll call this quantity "stuff". We want to generalize the continuity equation from one space dimension to two (or more) space dimensions.

There's no problem in generalizing the stuff density ρ . In two dimensions density will be a function $\rho(x_1, x_2, t)$ with units of stuff per area.

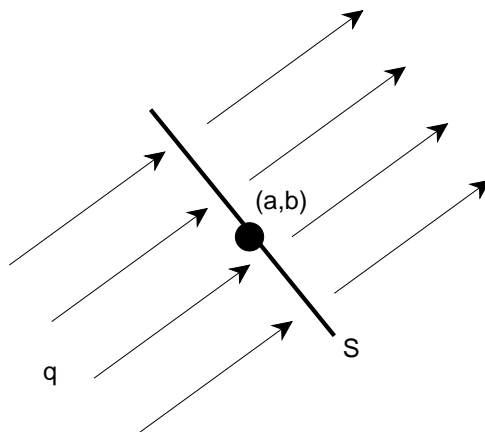
Generalizing the notion of flux requires slightly more care. In our case the flux will be a vector-valued quantity $\mathbf{q}(x_1, x_2, t) = \langle q_1(x_1, x_2, t), q_2(x_1, x_2, t) \rangle$. The function \mathbf{q} is an example of a *vector-field* (in this case time-dependent). For each time t and spatial point (x_1, x_2) , \mathbf{q} returns a two-dimensional vector. The idea is that \mathbf{q} captures both flow-rate and direction, but we need to be just a bit careful in quantifying this.

At a fixed time t the vector field \mathbf{q} might look like



You might be tempted to think of \mathbf{q} as the velocity of the stuff, but this is NOT generally true (except in special cases). Recall that in one dimension the rate-of-flow $q(x, t)$ was simply the amount of stuff per unit time flowing past the point x at time t . This isn't the same as the velocity of the stuff, e.g., velocity has dimension length per time, while q has dimension stuff (Kg or Coulombs, maybe) per time. How is the flow rate defined in two-dimensions? What dimensions should it have?

Consider a vector field \mathbf{q} like that shown in Figure 1. If the vector field is well-behaved (smooth enough, i.e., continuous, or better yet, differentiable in both time and space) then in the vicinity of any given point the vector field is nearly constant, so the vectors all have nearly the same length and direction. For example, at some specific time t near a general point (a, b) in Figure 1, the vector field might look like



The vector field in this picture is essentially constant, and equal to $\mathbf{q}(a, b, t)$. Consider, as pictured above, a short line segment S passing through the point (a, b) . The segment S is oriented so that it is perpendicular to the local flow. It is intuitively clear that if \mathbf{q} represents the rate of flow of stuff in the plane then the amount of stuff flowing across S should be jointly proportional to the magnitude of the vector $\mathbf{q}(a, b, t)$ and to $|S|$, the length of S ; if we doubled the length of S , twice as much stuff would flow over it, and if we double the flow rate \mathbf{q} we'd double the amount of stuff flowing over S per unit time. On dimensional grounds, then, we need \mathbf{q} to have dimensions such that \mathbf{q} times length has units of stuff per unit time, so \mathbf{q} has units of stuff per length per unit time. The vector $\mathbf{q}(a, b, t)$ is called the *flux* at the point (a, b) , time t .

More precisely, suppose some stuff flows around in the x_1x_2 plane. We'll define the flux $\mathbf{q}(a, b, t)$ at some point (a, b) and time t as follows: Let S be a "short" line segment of length ϵ with midpoint (a, b) and unit normal vector \mathbf{n} . Rotate S (and so rotate \mathbf{n}) until the flux of stuff across S in the direction \mathbf{n} is maximized. Call this particular line segment $S(\epsilon)$, it's normal vector $\mathbf{n}(\epsilon)$, and the resulting maximum flux $F(\epsilon)$. We define $\mathbf{q}(a, b, t)$ as

$$\mathbf{q}(a, b, t) = \lim_{\epsilon \rightarrow 0} \frac{F(\epsilon)}{S(\epsilon)} \mathbf{n}(\epsilon).$$

Exercises:

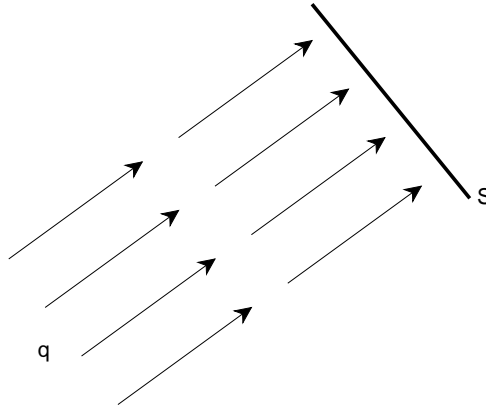
1. If \mathbf{q} represents the mass flux of some fluid flowing in the plane, what are reasonable MKS dimensions for \mathbf{q} ?
2. If \mathbf{q} represents an electric current flux in the plane, what are reasonable MKS dimensions for \mathbf{q} ?
3. If \mathbf{q} represents the flow of heat energy, what are reasonable MKS dimensions for \mathbf{q} ?

Flux Across a Surface

In one dimension it was easy, given a flux function q , to figure out how much "stuff" flows into and out of a one dimension interval. If the interval has endpoints at $x = a$ and $x = b$ then at a fixed time t the flux in is $q(a, t)$ and the flux out is $q(b, t)$, if positive q denotes left to right flow. The net input flux at time t is then $q(a, t) - q(b, t)$. This was a necessary ingredient in deriving the continuity equation for conservation of stuff. In two or more dimensions things get more complicated. We need to figure out how to quantify the rate at which stuff flows into or out of some region, or more generally, across some curve or surface.

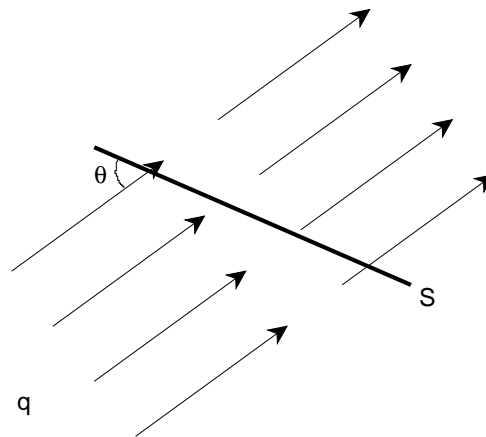
The following discussion will lead us to the answer, and is a perfect illustration of how one goes about setting up integrals in real problems. In what follows we'll simply assume that \mathbf{q} is a function of x and y only, since even for time-dependent fluxes we want to compute the flux only at fixed moments in time.

1. Below is a *constant* vector field \mathbf{q} ; think of it as the flux of some "stuff" flowing across the plane. The flow is perpendicular to the segment S .



If the length of the line segment S is denoted by $|S|$ and the magnitude of the flux is $|\mathbf{q}|$, it's easy to see that the amount of stuff crossing S per unit time is just $|S||\mathbf{q}|$, just as we discussed above.

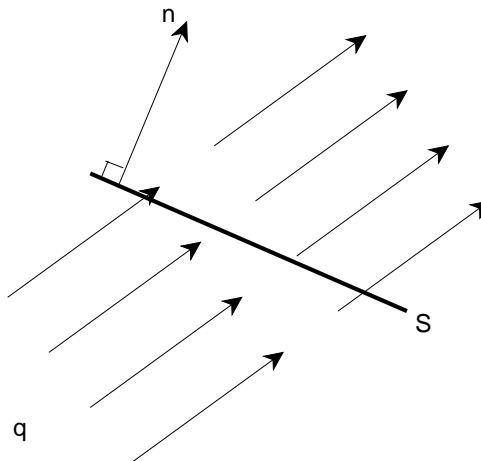
- Suppose now that the flux or vector field is still constant, but at angle θ to S :



It's not hard to see that the rate at which stuff is crossing S is just $|S||\mathbf{q}| \sin(\theta)$.

- Another way to write the flux across the line segment S is in terms of vector operations. Suppose that \mathbf{n} is a unit vector which is normal to

S , as pictured below.



If \mathbf{q} lies at an angle θ to S then \mathbf{q} lies at angle $\pi/2 - \theta$ to \mathbf{n} . Thus the flux is just $|S||bfq|\sin(\theta) = |S||\mathbf{q}|\cos(\pi/2 - \theta)$. But this last quantity is just $|S|\mathbf{q} \cdot \mathbf{n}$. So the flux is

$$\text{flux} = |S|\mathbf{q}(a, b) \cdot \mathbf{n}. \quad (1)$$

This is the flux of a constant \mathbf{q} over a straight line segment S .

4. We now have the means to compute the flux over any curve C , and it's just the type of reasoning used in Calc II. Given a curve C parameterized by $x_1 = x_1(s), x_2 = x_2(s)$ for $A \leq s \leq B$, we do the following: Partition the interval $A \leq s \leq B$ into n short subintervals (s_i, s_{i+1}) for $0 \leq i \leq n - 1$, with $s_0 = A$ and $s_n = B$. Consider that small part dC_i of the curve C parameterized by $(x_1(s), x_2(s))$ for $s_i \leq s \leq s_{i+1}$. If $s_{i+1} - s_i$ is small (and the curve sufficiently smooth) then dC is approximately a line segment, of length

$$dS_i = \sqrt{(x_1(s_{i+1}) - x_1(s_i))^2 + (x_2(s_{i+1}) - x_2(s_i))^2}. \quad (2)$$

But from the tangent line approximation (or just definition of the derivative) we have $x_1(s_{i+1}) - x_1(s_i) = (s_{i+1} - s_i)x_1'(s_i) + O((s_{i+1} - s_i)^2)$, at least if x_1 is twice-continuously differentiable. Similarly $x_2(s_{i+1}) - x_2(s_i) = (s_{i+1} - s_i)x_2'(s_i) + O((s_{i+1} - s_i)^2)$. Plug this information into

equation (2) to find

$$dS_i = \sqrt{(x'_1(s_i))^2 + (x'_2(s_i))^2}(s_{i+1} - s_i) + O((s_{i+1} - s_i)^2). \quad (3)$$

We're going to drop the $O((s_{i+1} - s_i)^2)$ now (it will go away in the limit we're about to take).

If $s_{i+1} - s_i$ is small then the curve dC_i is short and \mathbf{q} is approximately constant near dC_i , and in fact about equal to $\mathbf{q}(x_1(s_i), x_2(s_i))$. The flux over dC_i is thus (to second order)

$$dF_i = \mathbf{q}(x_1(s_i), x_2(s_i)) \cdot \mathbf{n}(s_i) dS_i. \quad (4)$$

All we need now is an explicit expression for the unit normal vector \mathbf{n} in terms of the parameterization.

Recall that for a parametric curve $x_1 = x_1(s), x_2 = x_2(s)$, the vector $\langle x'_1(s), x'_2(s) \rangle$ is tangent to the curve. It's easy to check that the vector $\langle x'_2(s), -x'_1(s) \rangle$ is normal to the curve (it's the 90 degree clockwise rotation of the tangent vector). We can make it a unit vector by dividing it by its own length, and so we can take

$$\mathbf{n}(s_i) = \frac{\langle x'_2(s_i), -x'_1(s_i) \rangle}{\sqrt{(x'_1(s_i))^2 + (x'_2(s_i))^2}} \quad (5)$$

As the unit normal vector to C at $s = s_i$ (and also on dC_i , approximately). If we plug this expression for \mathbf{n} into equation (4) and use equation (3) we obtain, after dropping all but the most significant terms,

$$dF_i = [\mathbf{q}(x_1(s_i), x_2(s_i)) \cdot \langle x'_2(s_i), -x'_1(s_i) \rangle](s_{i+1} - s_i). \quad (6)$$

The total flux over the curve C is approximately

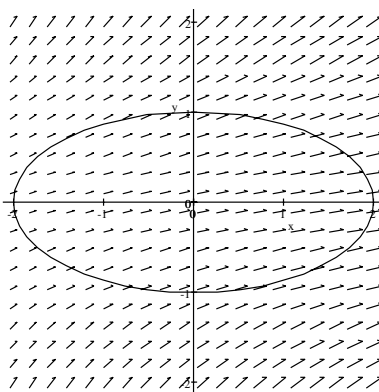
$$\sum_{i=0}^{n-1} dF_i.$$

If we let $n \rightarrow \infty$ the sum above becomes, by the usual Riemann sum process, the "true" flux over C , written $\int_C \mathbf{q} \cdot \mathbf{n} dS$, and given explicitly by

$$\int_C \mathbf{q} \cdot \mathbf{n} dS = \int_A^B (q_1(x_1(s), x_2(s))x'_2(s) - q_2(x_1(s), x_2(s))x'_1(s)) ds. \quad (7)$$

Example

Consider a curve S in the plane defined by $x_1 = 2 \cos s$, $x_2 = \sin s$, $0 \leq s < 2\pi$. Note that the normal vector field \mathbf{n} defined by equation (5) really does point outward. Suppose that the flow of some fluid in the plane is described by the vector field $\mathbf{q}(x_1, x_2) = (x_1 + 5)\mathbf{i} + (x_2^2 + 1)\mathbf{j}$, pictured below:



By formula (7) the total flux of \mathbf{q} over the curve S is given by

$$\int_0^{2\pi} [(2 \cos(s) + 5) \cos(s) + (\sin^2(s) + 1)(2 \sin(s))] ds = 2\pi.$$

(Note that wherever x_1 appears in the integral, it is replaced by $2 \cos s$, and similarly for x_2). This is the net OUTWARD flux (in the direction of \mathbf{n}).