Banach Space Valued Functions

MA 466
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Analytic Functions; Series

Let f be a function from $\mathbb{C}$ to a Banach Space B. We assume that B is a Banach space over $\mathbb{C}$, so that each element of B can be sensibly multiplied by any complex number. In this handout we'll be interested in extending some familiar results (especially results from complex analysis) to these types of functions.

Recall that a function f from $\mathbb{C}$ to $\mathbb{C}$ is said to be *analytic* at z_0 if f has a power series expansion which converges to f in some neighborhood of z_0, so

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j. \tag{1}$$

Here the a_j are complex numbers (though if f is a real-valued function of a real variable the a_j are of course real numbers, and f is said to be "real-analytic"). It turns out that a f defined by a series like (1) must be infinitely differentiable and in fact $a_j = f^{(j)}(z_0)/j!$. (You may recall there are other definitions of analyticity, all equivalent to this).

We will say that a function $f : \mathbb{C} \to B$ is analytic at z_0 if f has a power series expansion which converges to f in some neighborhood of z_0, so

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j \tag{2}$$

where here the a_j are elements of B. Note that the expression on the right in equation (2) has some chance of making sense: since $(z - z_0)^j$ is a complex number we can form the products $a_j(z - z_0)^j$ as elements of B and then add them. Plugging in $z = z_0$ shows that $a_0 = f(z_0)$.

Recall that by the root test the series in equation (1) converges if $|z - z_0| < R$ where

$$R = \limsup_{n\to\infty} |a_j|^{1/n} \tag{3}$$

since the series can then be dominated by a geometric series. EXACTLY the same reasoning shows that the series in (2) converges for $|z - z_0| < R$, where R is as in equation (3) but with $|a_j|$ replaced by $\|a_j\|$, the Banach space norm. Like the more elementary case, the essence of the proof is that since the series (3) is dominated by a geometrically convergent series, the partial sums form a Cauchy sequence in B. Since B is complete, the series converges.

Integration

What does it mean to integrate a function which takes values in a Banach space? First, suppose that $f : \mathbb{R} \to B$; it doesn't matter if f is analytic here. To integrate f over a bounded interval $(a, b) \subset \mathbb{R}$ take a partition P of the interval, say $P = \{a, x_1, x_2, \cdots, x_{n-1}, b\}$ and form a Riemann sum

$$S(P, f) = \sum_{j=0}^{n-1} f(x_j^*)(x_{j+1} - x_j) \tag{4}$$

where $x_j^* \in [x_j, x_{j+1}]$. Note that the right side of equation (4) is an element of B. Limit the mesh of the partition to zero. If the Riemann sum converges (in B) in the appropriate sense (just like in real analysis) we'll say that f is integrable over $[a, b]$ and set

$$\int_a^b f(x)\, dx = \lim_{\|P\| \to 0} S(P, f).$$

The integral will converge if, for example, f is continuous. Of course the integral is just an element of B.

Lebesgue integration can also be used, but we won't need it here.

Now suppose f is a function from $\mathbb{C}$ to B (again, it doesn't matter yet if f is analytic). We can also integrate over contours (i.e., curves) in $\mathbb{C}$, just as one does in complex analysis. In fact, the way one actually computes a contour integral of f over a curve C is to parameterize C as $z = z(t)$ for $a \le t \le b$, then compute

$$\int_C f(z)\, dz = \int_a^b f(z(t)) z'(t)\, dt \tag{5}$$

as an integral over the real interval (a, b) (though the integral contains a real and imaginary part; integrate them separately). Precisely the same formula works to evaluate a Banach space valued function over a contour in $\mathbb{C}$.

Recall Cauchy's integral formula, which says that if $f : D \to \mathbb{C}$ is analytic on an open region D and $z_0 \in D$ then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz$$

for any simple closed contour $C \subset D$ containing z_0. If z_0 is not inside C the integral is zero.

Consider a function $f : D \to B$ which is analytic in some region $D \subset \mathbb{C}$ with $z_0 \in D$, i.e., f has a power series of the form (2) which converges to f near z_0. We have

$$\begin{aligned}
\int_C \frac{f(z)}{z - z_0}\, dz &= \int_C \sum_{j=0}^{\infty} \frac{a_j (z - z_0)^j}{z - z_0}\, dz \\
&= \sum_{j=0}^{\infty} a_j \int_C \frac{(z - z_0)^j}{z - z_0}\, dz \\
&= a_0 \int_C \frac{dz}{z - z_0}\, dz + \sum_{j=1}^{\infty} a_j \int_C (z - z_0)^{j-1}\, dz \\
&= a_0 \int_C \frac{dz}{z - z_0}\, dz \\
&= 2\pi i a_0. \qquad (6)
\end{aligned}$$

In the above I made a few manipulations that you can easily justify are valid—integrating the convergent power series term by term (just like in real or complex analysis) and pulling “constants” like a_j outside the integrals. Also, the contour integrals of $(z - z_0)^k$ vanish if $k \geq 0$, since the integrand is analytic.

Of course since $a_0 = f(z_0)$ equation (6) yields

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz, \qquad (7)$$

the Cauchy integral formula. It holds even for analytic functions which take values in a Banach space!

Liouville’s Theorem

Liouville’s Theorem from complex analysis says that an entire function (analytic on the whole complex plane) which is bounded must be constant. You prove it by differentiating both sides of equation (7) with respect to z_0 (permissible!) to find

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^2}\, dz. \qquad (8)$$

Take the contour to be a ball of radius r about z_0 and let $r \to \infty$. Since f is bounded it's easy to see that the right side of equation (8) goes to zero, i.e., $f'(z_0) = 0$ for all z_0, so f is constant.

Precisely the same proof works if f takes values in a Banach space!

Laurent Series

Let $f : \mathbb{C} \to \mathbb{C}$ be a function which is analytic OUTSIDE a ball $B_r(0) \subset \mathbb{C}$. For simplicity suppose that $f(z) \to 0$ as $z \to \infty$. Then f has a Laurent expansion of the form

$$f(z) = \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \tag{9}$$

for all $|z| > r$. The reason is this: Let $g(z) = f(1/z)$, so g is analytic in a disk $B_R(0)$; since f tends to zero as z approaches infinity, g has a removable singularity at $z = 0$ (set $g(0) = 0$). Thus g really is analytic in the ball $B_R(0)$, and so has a convergent series expansion

$$g(z) = a_1 z + a_2 z^2 + \cdots \tag{10}$$

for $|z| < R$. Since $f(z) = g(1/z)$, we immediately obtain equation (9) for $|z| > r = 1/R$. That the expansion is unique follows from the fact that the expansion for g is unique.

Exactly the same reasoning works if f takes values in a Banach space. If a function f is analytic outside some disk R then f has a unique Laurent expansion of the from (9), where the a_j are of course elements of the Banach space.