Completeness, Eigenvalues, and the Calculus of Variations

MA 436
Kurt Bryan

1 Introduction

Our goal here is to prove that $\phi_k(x) = \sqrt{2}\sin(k\pi x)$ for $k = 1, 2, 3, \ldots$ form a complete orthogonal family of functions on the interval $(0, 1)$; we already know they're orthonormal—it's completeness that's hard. With obvious modifications everything we will prove is also true on any interval (a, b) with a and b finite. In the course of proving that this family is complete we'll examine an important area of PDE known as *eigenvalue* problems, and develop some simple ideas from a branch of mathematics known as the *calculus of variations*. This also forms the mathematical basis of *finite element methods*.

The crucial observation in proving that the ϕ_k are complete is to note that these functions are non-zero solutions to the differential equation

$$\phi''(x) + \lambda\phi(x) = 0 \tag{1}$$

with $\phi(0) = \phi(1) = 0$ and $\lambda = k^2\pi^2$. In fact, if you look back at the solution procedure for the wave equation on the interval $(0, 1)$ you'll see that equation (1) came straight out of the separation of variables process. The fact that $\lambda = k^2\pi^2$ was forced by the boundary conditions on ϕ, at least if you want ϕ to be non-zero.

Some terminology: If a number λ is such that equation (1) has a non-zero solution with zero boundary conditions, then λ is called an *eigenvalue* for $\frac{d^2}{dx^2}$. We already know that the eigenvalues are $\lambda_k = k^2\pi^2$ for $k = 1, 2, 3, \ldots$. The corresponding function ϕ_k that solves (1) for $\lambda = \lambda_k$ is called the *eigenfunction* for λ_k. We already know that the eigenfunctions here are $\phi_k(x) = \sin(k\pi x)$, or really any multiple of $\sin(k\pi x)$.

The first step in proving completeness is to recast equation (1) in a different form, one that looks totally unrelated to a differential equation. Consider the problem of finding the smallest eigenvalue, $\lambda_1 = \pi^2$, and its eigenfunction $\sin(\pi x)$. I claim that this is equivalent to minimizing the quantity

$$Q(\phi) = \frac{\int_0^1 (\phi'(x))^2\,dx}{\int_0^1 \phi^2(x)\,dx}$$

over all $C^1(0,1)$ functions with $\phi(0) = \phi(1) = 0$. This is like a Calc 3 optimization problem, except that

- The object Q being minimized is not a function, but an integral or ratio of integrals, and
- The input to Q is not a number or set of numbers, but a function.

Before we can show that minimizing Q is equivalent to finding eigenvalues, we need to know more about the calculus of variations.

1.1 The Calculus of Variations

The calculus of variations is about finding functions that minimize integrals. This might sound like a useless problem, but it's actually one of the most important areas of classical and modern applied math. The solutions to most PDE's can be found by a minimization process, and this leads to finite element methods. This is also the basis of much of optimal control theory, and the calculus of variations is part of the mathematics behind the Lagrangian and Hamiltonian formulations of the laws of physics.

Let's consider a simple model problem. We want to find a function which minimizes the integral

$$Q(\phi) = \int_0^1 (\phi'(x))^2 \, dx \tag{2}$$

with the additional restriction that $\phi(0) = 2$ and $\phi(1) = 4$. We should clearly consider only differentiable (say $C^1([0,1])$) functions. How do we go about doing this?

Pretend that you're a junior in high school again; you've studied precalculus mathematics, so you know what functions are, but you've never seen a derivative. Someone asks you to find a positive number x_0 which minimizes the function $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + 3$, at least locally. You know what that means: the minimizer x_0 will be a number which is lower than any nearby number. In other words

$$f(x_0) \leq f(x_0 + \epsilon)$$

for all sufficiently small ϵ. If you expand the above inequality out and do some simple algebra you obtain

$$(x_0^2 + x_0 - 2)\epsilon + (x_0 + \frac{1}{2})\epsilon^2 + \frac{1}{3}\epsilon^3 \geq 0 \tag{3}$$

where the powers of ϵ have been grouped together. Think of x_0 as fixed and consider the left side of inequality (3) as a function of ϵ. How can the left side be positive for ALL ϵ? If $|\epsilon|$ is sufficiently close to zero then the ϵ^2 and higher powers are negligible compared to ϵ; the sign of the left side will be determined by ϵ in this case. If $x_0^2 + x_0 - 2 > 0$ then choosing $\epsilon < 0$ and small will violate the inequality (3). If $x_0^2 + x_0 - 2 < 0$ then choosing $\epsilon > 0$ and small will again violate the inequality (3). The only way that (3) can hold for all ϵ is if $x_0^2 + x_0 - 2 = 0$, so that the ϵ^2 term dominates. But if $x_0^2 + x_0 - 2 = 0$ then we can solve for the positive root and find $x_0 = 1$. This is where the minimum must be.

The same procedure allows us to find the minimizer of $Q(\phi)$. Let's use f to denote that function which minimizes Q. Just as we perturbed x_0 above, by adding in a small number ϵ, we will perturb the minimizer f by adding in a small function. Let $\eta(x)$ be any nice differentiable function with $\eta(0) = \eta(1) = 0$. Then if f really is the minimizer of Q (with the right boundary conditions) then

$$Q(f + \epsilon\eta) \geq Q(f) \tag{4}$$

for ANY small number ϵ, and ANY function η with $\eta(0) = \eta(1) = 0$. Why must we require that $\eta(0) = \eta(1) = 0$? Because otherwise the function $f(x) + \epsilon\eta(x)$ doesn't satisfy the boundary conditions and so wouldn't be a legitimate contender for the minimizer. Expand out both sides of (4) and do a bit of algebra to find that

$$2\epsilon \int_0^1 f'(x)\eta'(x)\,dx + \epsilon^2 \int_0^1 (\eta'(x))^2\,dx \geq 0. \tag{5}$$

How is it possible for the left side to be non-negative for ALL choices of ϵ? Only if the coefficient of ϵ is zero, for otherwise we could violate the inequality. We conclude that

$$\int_0^1 f'(x)\eta'(x)\,dx = 0. \tag{6}$$

Integrate equation (6) by parts with $u = f$ and $dv = \eta' dx$. With the boundary conditions on η we obtain

$$\int_0^1 f''(x)\eta(x)\,dx = 0. \tag{7}$$

Note that we just implicitly imposed the requirement that the minimizer f has a second derivative which is differentiable, e.g., $f \in C^2([0,1])$. The function η was ANY differentiable function with $\eta(0) = \eta(1) = 0$. How can equation (7) be true for any such η? You should be able to convince yourself that this can hold ONLY IF $f''(x) = 0$ on the interval $(0,1)$. But this means that $f(x)$ is of the form $f(x) = c_1 x + c_2$, and if it's to satisfy $f(0) = 2$ and $f(1) = 4$ then f must be $f(x) = 2x + 2$.

WARNINGS: What we've really done above is to show that IF a minimizer f for Q as defined by equation (2) exists, and IF this minimizer has an integrable second derivative, THEN $f(x) = 2x + 2$. It's conceivable that:

1. Another function which is, say, just C^1 might make Q smaller. Think of an analogous situation from calculus: If we minimize a function f over an interval $[a,b]$, that doesn't mean there might not be a number outside $[a,b]$ which makes f smaller.

2. No function minimizes Q. An analogous situation in calculus would be minimizing $f(x) = x$ over the OPEN interval $0 < x < 1$.

In what follows we're generally going to assume that the minimizer f exists—this will usually be highly plausible, if not proven—and we'll then find the minimizer f subject to certain conditions. Typically we'll require f to be at least twice-differentiable.

Sometimes, though, you really can see that the function f found by the above procedure is the best possible. In the last example, with $f(x) = 2x+2$, we see that

$$Q(f + \epsilon\eta) - Q(f) = \epsilon^2 \int_0^1 (\eta'(x))^2 \, dx \geq 0$$

for all $\eta \in C^1([0,1])$ with $\eta(0) = \eta(1) = 0$, so f is guaranteed to beat any $C^1([0,1])$ function.

Let's look at a few more examples:

Example 2: Consider minimizing

$$Q(\phi) = \int_0^1 ((\phi'(x))^2 + \phi^2(x)) \, dx$$

with the conditions that $\phi(0) = 1$, but with no other restrictions on ϕ. The minimizer f will satisfy

$$Q(f) \leq Q(f + \epsilon\eta)$$

for all C^1 functions η with $\eta(0) = 0$ and real numbers ϵ. Note here that because we have no restrictions at the endpoint $x = 1$, there are no restrictions on $\eta(1)$. If you expand out the above equation you find that

$$\epsilon \left(2 \int_0^1 (f(x)\eta(x) + f'(x)\eta'(x))\, dx \right) + O(\epsilon^2) \geq 0.$$

Reasoning as before, this can hold only if

$$\int_0^1 (f(x)\eta(x) + f'(x)\eta'(x))\, dx = 0.$$

Integrate the second term by parts and use the fact that $\eta(0) = 0$ to find that

$$\int_0^1 (f(x) - f''(x))\eta(x)\, dx + f'(1)\eta(1) = 0$$

for all functions η with $\eta(0) = 0$. Now reason as before: if $f(x) - f''(x) \neq 0$ for some x in $(0, 1)$ then we can certainly find a function η (and can arrange $\eta(1) = 0$, too) so that the integral above is not zero. We conclude that $f - f''$ is identically zero on the interval $(0, 1)$. If this is true then the above equation becomes $f'(1)\eta(1) = 0$, and since we can arrange for $\eta(1)$ to be anything we like, we must conclude that $f'(1) = 0$. So in the end the minimizer f must satisfy

$$f(x) - f''(x) = 0$$

with $f(0) = 1$ and $f'(1) = 0$. The general solution to $f - f'' = 0$ is $f(x) = c_1 e^x + c_2 e^{-x}$. To arrange the boundary conditions we solve two equations in two unknowns, $c_1 + c_2 = 1$, $c_1 e - c_2/e = 0$ to find $c_1 = 1/(e^2 + 1)$, $c_2 = e^2/(e^2 + 1)$. The minimizer is

$$f(x) = \frac{1}{e^2 + 1} \left(e^x + e^{2-x} \right).$$

Example 3: Let's prove that the shortest path between two points is a straight line! Let the points be (x_0, y_0) and (x_1, y_1). Let's also make the assumption that the path can be described as the graph of a function $y = f(x)$. Then what we seek to minimize is

$$Q(\phi) = \int_{x_0}^{x_1} \sqrt{1 + (\phi'(x))^2}\, dx$$

subject to the restrictions that $\phi(x_0) = y_0$ and $\phi(x_1) = y_1$. As before, we know that the shortest path f will satisfy $Q(f) \leq Q(f + \epsilon\eta)$ where $\eta(x_0) = 0$ and $\eta(x_1) = 0$, and ϵ is any number. Written out in detail this is just

$$0 \leq \int_{x_0}^{x_1} (\sqrt{1 + (f'(x))^2 + 2\epsilon f'(x)\eta'(x) + \epsilon^2(\eta'(x))^2} - \sqrt{1 + (f'(x))^2})\, dx. \quad (8)$$

Looks like a mess. We can simplify by using the fact that if ϵ is a small number then

$$\sqrt{a + \epsilon b} = \sqrt{a} + \frac{b}{2\sqrt{a}}\epsilon + O(\epsilon^2).$$

This comes from the tangent line or Taylor series approximation. Apply this to the integrand in inequality (8) with $a = 1 + (f'(x))^2$ and $b = 2f'\eta' + \epsilon(\eta')^2$. You find that the first messy square root is

$$\sqrt{1 + (f'(x))^2 + 2\epsilon f'(x)\eta'(x) + \epsilon^2(\eta'(x))^2} = \sqrt{1 + (f'(x))^2} + \epsilon\frac{f'\eta'}{\sqrt{1 + (f')^2}} + O(\epsilon^2).$$

Put this into the integral in (8), do the obvious cancellations, and note that as before we need the ϵ term to be zero if the inequality is to hold up. This gives

$$\int_{x_0}^{x_1} \frac{f'(x)\eta'(x)}{\sqrt{1 + (f'(x))^2}}\, dx = 0$$

for all functions η with $\eta(x_0) = \eta(x_1) = 0$. Integrate by parts in x to get the derivative terms off of η. It's messy, but in the end you get (using the endpoint conditions on η)

$$\int_{x_0}^{x_1} \eta(x)\frac{f''(x)}{(1 + (f'(x))^2)^{3/2}}\, dx = 0.$$

As before, for this to be true we need $\frac{f''(x)}{(1+(f'(x))^2)^{3/2}} = 0$. The denominator is always positive, so for this to be true we need $f''(x) = 0$, i.e., $f(x) = mx + b$, a line. Of course, m and b are adjusted to make it go through the two points (x_0, y_0) and (x_1, y_1).

Example 4: Minimize

$$Q(\phi) = \int_0^1 (\phi'(x))^2\, dx$$

subject to $\phi(0) = 1$, $\phi(1) = 0$, and the condition that

$$\int_0^1 \phi(x)\sin(\pi x)\,dx = 0$$

so ϕ is orthogonal to $\sin(\pi x)$. To solve this let's first recall the notation $(f,g) = \int fg$. Also, let's use $\phi_1(x)$ to denote the function $\sqrt{2}\sin(\pi x)$.

Let the minimizer be called f; note that $(f,\phi_1) = 0$. Normally we'd replace ϕ in $Q(\phi)$ by $f + \epsilon\eta$, where η is some function with $\eta(0) = \eta(1) = 0$, so $f + \epsilon\phi$ satisfies the boundary conditions. But if we do this here the function $f + \epsilon\phi$ won't generally satisfy $(f + \epsilon\eta, \phi_1) = 0$. However, consider taking η according to

$$\eta = \psi - c\phi_1,$$

where ψ is an arbitrary function with $\psi(0) = \psi(1) = 0$ and $c = (\psi, \phi_1)$. It's easy to check that $\eta(0) = \eta(1) = 0$, and that $(\eta, \phi_1) = 0$, for ANY choice of ψ. This η is a legitimate choice to put into $Q(f) \leq Q(f + \epsilon\eta)$. The usual argument—expand powers, cancel like terms, set the first variation equal to zero—leads to (with $\eta = \psi - c\phi_1$)

$$\int_0^1 f'(x)(\psi'(x) - c\phi_1'(x))\,dx = 0.$$

Integrate the first term, $f'\psi'$, by parts to get all derivatives onto f; integrate the second term by parts to get all derivatives onto ϕ_1. With the boundary conditions $\psi(0) = \psi(1) = \phi_1(0) = \phi_1(1) = 0$ you obtain

$$-\int_0^1 f''(x)\psi(x)\,dx - c\left(f(x)\phi_1'(x)|_{x=0}^{x=1} - \int_0^1 f(x)\phi_1''(x)\,dx\right) = 0.$$

Now use the fact that $\phi_1'' = -\pi^2\phi_1$ (how convenient!) in the last integral, with the facts that $(f,\phi_1) = 0$, $\phi'(0) = \pi\sqrt{2}, \phi'(1) = -\pi\sqrt{2}$, to obtain

$$-\int_0^1 f''(x)\psi(x)\,dx + c\pi\sqrt{2}f(1) + c\pi\sqrt{2}f(0) = 0 \tag{9}$$

for ANY function ψ, where $c = (\psi, \phi_1)$. What can we deduce from this? With the conditions $f(0) = 1$, $f(1) = 0$, this shows immediately that

$$-\int_0^1 f''(x)\psi(x)\,dx + c\pi\sqrt{2} = 0.$$

Use $c = (\psi, \phi_1)$ in the above equation to find that

$$\int_0^1 \psi(x)(\pi\sqrt{2}\phi_1(x) - f''(x))\, dx = 0$$

for all ψ. This forces $f''(x) = \pi\sqrt{2}\phi_1(x) = 2\pi\sin(\pi x)$. Integrate twice in x to find that $f(x) = -\frac{2}{\pi}\sin(\pi x) + c_1 x + c_2$. To obtain $f(0) = 1$ and $f(1) = 0$ we need $c_1 = -1$ and $c_2 = 1$. The minimizer is

$$f(x) = -\frac{2}{\pi}\sin(\pi x) - x + 1.$$

General Principles: The previous examples lead us to some general principles for solving calculus of variations problems. I don't want to give a very specific recipe, just some general guidelines. To minimize $Q(\phi)$.

1. Start with the inequality

   $$Q(f) \leq Q(f + \epsilon\eta)$$

   where f is the minimizer to be found and η is a legitimate test function to put into Q.

2. One way or another, manipulate the inequality above into something of the form

   $$0 \leq \epsilon V_1 + O(\epsilon^2)$$

   where V_1 is typically an integral involving f and η. The quantity V_1 is called the *first variation*. For the above inequality to hold for all ϵ we need $V_1 = 0$.

3. Manipulate V_1, often using integration by parts, to determine f. This usually leads to an equation like

   $$\int L(f)\eta\, dx = 0$$

   where $L(f)$ is some kind of differential operator applied to f. The only way this integral can be zero is for $L(f) = 0$. Find f by solving the DE. The initial or boundary conditions will generally be obvious.

1.2 Eigenfunctions via Calculus of Variations

We're almost in a position to prove that the functions $\phi_k(x) = \sin(k\pi x)$, $k = 1, 2, \ldots$, are complete on the interval $(0, 1)$. In what follows I won't put a $\sqrt{2}$ in front of the sine, as it won't matter. As mentioned above, the first crucial fact is that these functions are in fact eigenfunctions for d^2/dx^2 with zero boundary conditions, that is,

$$\frac{d^2\phi_k}{dx^2} + \lambda_k \phi_k = 0$$

with $\phi_k(0) = \phi_k(1) = 0$ and $\lambda_k = k^2\pi^2$. These functions and the associated λ_k are the only non-zero solutions to $\frac{d^2\phi}{dx^2} + \lambda\phi = 0$ with zero boundary conditions. In order to prove completeness we now cast the problem of finding the eigenfunctions into a calculus of variations problem.

Claim 1: The function $\sin(\pi x)$ (or any multiple of this function) is the minimizer of

$$Q(\phi) = \frac{\int_0^1 (\phi'(x))^2\,dx}{\int_0^1 \phi^2(x)\,dx} = \frac{(\phi', \phi')}{(\phi, \phi)} \tag{10}$$

and the minimum value is π^2.

Proof: Let a minimizer of Q be called $\phi_1(x)$; we need to show that $\phi_1(x)$ is a multiple of $\sin(\pi x)$. Let $\lambda_1 = Q(\phi_1)$. Obviously $\lambda_1 \geq 0$. We know that for any $C^1([0, 1])$ function $\eta(x)$ with $\eta(0) = \eta(1) = 0$ we have

$$\lambda_1 \leq Q(\phi_1 + \epsilon\eta)$$

or, if you write out the definition of Q,

$$\lambda_1(\phi_1 + \epsilon\eta, \phi_1 + \epsilon\eta) \leq (\phi_1' + \epsilon\eta', \phi_1' + \epsilon\eta').$$

Expand out the terms to obtain

$$\lambda_1(\phi_1, \phi_1) + 2\epsilon\lambda_1(\phi_1, \eta) + \epsilon^2\lambda_1(\eta, \eta) \leq (\phi_1', \phi_1') + 2\epsilon(\phi_1', \eta') + \epsilon^2(\eta', \eta').$$

But since $Q(\phi_1) = \lambda_1$ we have $\lambda_1(\phi_1, \phi_1) = (\phi_1', \phi_1')$. Cancelling these terms from both sides above and setting the first variation to zero shows that

$$\lambda_1(\phi_1, \eta) - (\phi_1', \eta') = 0.$$

Now explicitly write out the integrals,

$$\int_0^1 (\lambda_1 \eta(x)\phi_1(x) - \eta'(x)\phi_1'(x))\, dx = 0.$$

Do an integration by parts on the second term, to get the derivative off of η. With the boundary conditions on η we obtain

$$\int_0^1 \eta(x)(\lambda_1\phi_1(x) + \phi_1''(x))\, dx = 0,$$

By the usual argument we have

$$\lambda_1\phi_1(x) + \phi_1''(x) = 0.$$

Also remember we require $\phi_1(0) = \phi_1(1) = 0$. But we've already seen this! The only solutions are multiples of $\phi_1(x) = \sin(k\pi x)$ with $\lambda_1 = k^2\pi^2$ for integers k. But we want the solution that makes Q a minimum. You can easily check that $Q(\sin(k\pi x)) = k^2\pi^2$, so the minimum will occur with $k = 1$. Thus the minimizer of Q is any multiple of $\phi_1(x) = \sin(\pi x)$ with corresponding $\lambda_1 = \pi^2$.

Claim 2: The function $\sin(2\pi x)$ is the minimizer of $Q(\phi)$ with the restrictions $\phi(0) = \phi(1) = 0$ and the additional restriction that $(\phi, \phi_1) = 0$, where $\phi_1(x) = \sin(\pi x)$.

Proof: Let ϕ_2 denote the minimizer and let $\lambda_2 = Q(\phi_2)$. Then we know that

$$\lambda_2 \le Q(\phi_2 + \epsilon\eta)$$

where $\eta(0) = \eta(1) = 0$. But we also need $(\eta, \phi_1) = 0$. To achieve this let's take η to be of the form $\eta = \psi - c\phi_1$ where $c = (\psi, \phi_1)/(\phi_1, \phi_1)$ and ψ is any functions with $\psi(0) = \psi(1) = 0$. You can check that any such η is indeed orthogonal to ϕ_1. Then expanding the above inequality yields

$$\lambda_2(\phi_2, \phi_2) + 2\epsilon\lambda_2(\phi_2, \eta) + O(\epsilon^2) \le (\phi_2', \phi_2') + 2\epsilon(\phi_2', \eta') + O(\epsilon^2).$$

Substitute in $\eta = \psi - c\phi_1$ and note that $\lambda_2(\phi_2, \phi_2) = (\phi_2', \phi_2')$, so we can cancel these terms on both sides to get

$$2\lambda_2\epsilon(\phi_2, \psi) - 2c\lambda_2\epsilon(\phi_2, \phi_1) + O(\epsilon^2) \le 2\epsilon(\phi_2', \psi') - 2c\epsilon(\phi_2', \phi_1') + O(\epsilon^2).$$

We can pick out the first variation, the coefficient of ϵ—it must be zero for this inequality to hold, and so

$$\lambda_2(\phi_2, \psi) - c(\phi_2, \phi_1) - (\phi_2', \psi') + c(\phi_2', \phi_1') = 0. \tag{11}$$

But remember, we require that $(\phi_2, \phi_1) = 0$, so drop that term. Also, note that

$$\begin{aligned}
(\phi_2', \phi_1') &= \int_0^1 \phi_2'(x) \frac{d}{dx}(\sin(\pi x))\, dx, \\
&= -\int_0^1 \phi_2(x) \frac{d^2}{dx^2}(\sin(\pi x))\, dx, \\
&= \pi^2 \int_0^1 \phi_2(x) \sin(\pi x)\, dx, \\
&= \pi^2 (\phi_2, \phi_1), \\
&= 0.
\end{aligned}$$

where we integrated by parts to get the derivatives off of ϕ_2 and then used the fact that $\frac{d^2}{dx^2}(\sin(\pi x)) = -\pi^2 \sin(\pi x)$. All in all equation (11) becomes

$$\lambda_2(\phi_2, \psi) - (\phi_2', \psi') = 0$$

where ψ is any function with $\psi(0) = \psi(1) = 0$. But this is exactly what we got for ϕ_1, and it implies that $\phi_2'' + \lambda_2 \phi_2 = 0$ with zero boundary conditions. As before, the only non-zero solutions are $\phi_2(x) = \sin(k\pi x)$ with minimum value $Q(\phi_2) = \lambda_2 = k^2\pi^2$. But the requirement that $(\phi_2, \phi_1) = 0$ rules out taking $k = 1$. The next best we can do is $k = 2$, which yields $\phi_2(x) = \sin(2\pi x)$ with $\lambda_2 = 4\pi^2$.

Claim 3: The general claim is this: The nth eigenfunction $\sin(n\pi x)$ is obtained by minimizing $Q(\phi)$ with the requirement that $\phi(0) = \phi(1) = 0$ and $(\phi, \sin(k\pi x)) = 0$ for $k < n$. The minimum value is $\lambda_n = n^2\pi^2$.

Proof: I'll be brief! Let the minimum be called ϕ_n and let $\lambda_n = Q(\phi_n)$. Then

$$\lambda_n \leq Q(\phi_n + \epsilon\eta)$$

for any η with $\eta(0) = \eta(1) = 0$ and $(\eta, \phi_k) = 0$ for $k = 1, 2, \ldots, n-1$. We can take η to be of the form

$$\eta = \psi - \sum_{k=1}^{n-1} c_k \phi_k$$

where ψ is any function with $\psi(0) = \psi(1) = 0$ and $c_k = (\psi, \phi_k)/(\phi_k, \phi_k)$. Then you can easily check that $(\eta, \phi_k) = 0$ for $k = 1, 2, \ldots, n-1$. You have to use the fact that $(\phi_j, \phi_k) = 0$ if $j \neq k$ (the sine functions are orthogonal). Expand out Q in $\lambda_n \leq Q(\phi_n + \epsilon\eta)$, use the fact that $\lambda_n(\phi_n, \phi_n) = (\phi_n', \phi_n')$, and dump the $O(\epsilon^2)$ terms to obtain

$$(\phi_n, \psi) - \sum_{k=1}^{n-1} c_k(\phi_n, \phi_k) - (\phi_n', \psi') + \sum_{k=1}^{n-1} c_k(\phi_n', \phi_k') = 0. \tag{12}$$

But all the terms in the first sum are zero, since we require $(\phi_n, \phi_k) = 0$ for $k < n$. All the terms in the second sum die out, too, since

$$\begin{aligned}
(\phi_n', \phi_k') &= \int_0^1 \phi_n'(x) \frac{d}{dx}(\sin(k\pi x))\, dx, \\
&= -\int_0^1 \phi_n(x) \frac{d^2}{dx^2}(\sin(k\pi x))\, dx, \\
&= k^2\pi^2 \int_0^1 \phi_n(x) \sin(k\pi x)\, dx, \\
&= k^2\pi^2(\phi_n, \phi_k), \\
&= 0.
\end{aligned}$$

So all in all equation (12) is just

$$(\phi_n, \psi) - (\phi_n', \psi') = 0$$

the same as the case $n = 1$ and $n = 2$. We conclude that ϕ_n satisfies $\phi_n'' + \lambda_n \phi_n = 0$ with zero boundary conditions. The only choices are $\phi_n(x) = \sin(k\pi x)$ with $Q(\phi_n) = n^2\pi^2$, but the requirement $(\phi_n, \sin(k\pi x)) = 0$ for $k < n$ rules out the first $n-1$ choices. The best we can do is $k = n$ (remember, we're trying to make Q as small as possible). The minimizer is thus $\phi_n(x) = \sin(n\pi x)$ with $\lambda_n = Q(\phi_n) = n^2\pi^2$.

Summary: The eigenfunctions for the differential equation

$$\phi'' + \lambda\phi = 0$$

with boundary conditions $\phi(0) = \phi(1) = 0$ are $\phi_n(x) = \sin(n\pi x)$ with corresponding $\lambda_n = n^2\pi^2$. The eigenfunction ϕ_n can also be obtained as the minimizer of $Q(\phi)$ with the restriction $(\phi_n, \phi_k) = 0$ for $k < n$.

1.3 Completeness

We're ready! Remember (it's been so long) the idea behind completeness is the following. We want to take an $L^2(0,1)$ function $f(x)$ and expand it as

$$f(x) = \sum_{k=1}^{\infty} c_k \phi_k(x)$$

where $\phi_k(x) = \sqrt{2}\sin(k\pi x)$; note I put the $\sqrt{2}$ back, for orthonormality. As we've seen, we'll choose the c_k as

$$c_k = (f, \phi_k). \tag{13}$$

We want to show that the ϕ_k form a complete family, that is, for any $f \in L^2(0,1)$ we have

$$\lim_{n\to\infty} \int_0^1 \left(f(x) - \sum_{k=1}^{n} c_k \phi_k(x) \right)^2 dx = 0. \tag{14}$$

Actually, we're only going to prove that any function $f \in C^1([0,1])$ with $f(0) = f(1) = 0$ can be approximated like (14). Although it's true even if f doesn't vanish at the endpoints or isn't differentiable, it's easier to prove if we make this assumption. And it's not hard to use the C^1 result to prove the more general case.

Completeness Theorem: Any $C^1([0,1])$ function f with $f(0) = f(1) = 0$ can be approximated in the form (14).

Proof: The orthonormal part we've done. Let f be any $C^1([0,1])$ function. This means that f is automatically in $L^2(0,1)$. Define

$$f_n(x) = f(x) - \sum_{k=1}^{n-1} c_k \phi_k(x)$$

with c_k chosen according to equation (13). The functions $f_n(x)$ are the "remainders" after we've used an $n-1$ term sine approximation to f. You can easily check that $(f_n, \phi_j) = 0$ for $j < n$, for

$$(f_n, \phi_k) = (f - \sum_{j=1}^{n} c_j \phi_j, \phi_k) = (f, \phi_k) - c_k = 0$$

where we've used $(\phi_j, \phi_k) = 0$ if $j \neq k$ and the definition of c_k. Remember that $\phi_n(x) = \sqrt{2}\sin(n\pi x)$ minimizes $Q(\phi)$ subject to the restrictions $\phi(0) = \phi(1) = 0$ and $(\phi, \phi_k) = 0$ for $k < n$. The minimum value is $Q(\sqrt{2}\sin(n\pi x)) = \lambda_n = n^2\pi^2$. The function f_n satisfies these restrictions—it's a legitimate "candidate" to minimize Q—so that

$$\lambda_n \leq Q(f_n)$$

or

$$\lambda_n(f_n, f_n) \leq (f_n', f_n'). \tag{15}$$

Claim: For all n, $(f_n', f_n') \leq (f', f')$, so that (f_n', f_n') is bounded in n.

Proof of Claim: It's just a simple computation.

$$\begin{aligned}(f_n', f_n') &= (f' - \sum c_k\phi_k', f' - \sum c_k\phi_k'), \\ &= (f', f') - 2\sum c_k(f', \phi_k') + (\sum c_j\phi_j', \sum c_k\phi_k'). \end{aligned}\tag{16}$$

You can integrate by parts and use $\phi_k'' = -\lambda_k\phi_k$ to see that

$$(f', \phi_k') = -(f, \phi_k'') = \lambda_k(f, \phi_k) = \lambda_k c_k.$$

A similar integration by parts shows that

$$(\phi_j', \phi_k') = -(\phi_j, \phi_k'') = \lambda_k(\phi_j, \phi_k)$$

which is 0 if $j \neq k$ and λ_k if $j = k$. With these above two equations we can simplify equation (16) as

$$(f_n', f_n') = (f', f') - \sum_{k=1}^{n-1} \lambda_k c_k^2.$$

This makes it clear that $(f_n', f_n') \leq (f', f')$ for all n, since $\lambda_k > 0$ for all k.

With this claim inequality (15) yields

$$\lambda_n(f_n, f_n) \leq (f', f')$$

or

$$(f_n, f_n) \leq \frac{(f', f')}{\lambda_n}. \tag{17}$$

Since (f', f') is finite and independent of n, and since $\lambda_n = n^2\pi^2$, it's obvious that the right side of the above inequality approaches zero as n get large, so that

$$\lim_{n\to\infty}(f_n, f_n) = 0.$$

But since $f_n = f - \sum_{k=1}^{n-1} c_k\phi_k$ this is EXACTLY equation (14), the definition of completeness!

Slightly Challenging Problem: Suppose $f \in C^1([0,1])$ but $f(0)$ and/or $f(1)$ are not necessarily zero. Show that the Fourier series for f still converges to f in L^2.

Slightly More Challenging Problem: Suppose f is only piecewise C^1 (that is, the interval $[0,1]$ can be partitioned into intervals $[a_i, b_i]$, overlapping only at endpoints, and f is C^1 on each interval). Show that the Fourier series for f still converges to f in L^2.

1.4 Remarks

These ideas generalize to other differential equations and other orthogonal families of functions. A more general version of the differential equation $f'' + \lambda f = 0$ is the equation

$$\frac{d}{dx}(p(x)f'(x)) - q(x)f'(x) + \lambda r(x)f(x) = 0 \tag{18}$$

where p, q, and r are specified functions (with a few restrictions). On any interval (a, b) with boundary conditions $f(a) = 0$, $f(b) = 0$, it turns out that there is a non-zero solution to the DE only for certain values of λ. In fact, these values form a set of eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots$ with corresponding eigenfunctions ϕ_1, ϕ_2, $\phi_3, \ldots$. Using essentially the same techniques we used above it can be shown that the ϕ_k form a complete set. The main technical difficulties arise from the fact that we can't generally write down the functions ϕ_k explicitly (unlike our case with sines and cosines) and we can't write down the λ_k explicitly either. One crucial property of the λ_k that we used was that $\lambda_k \to \infty$ as $k \to \infty$. Without an explicit formula for λ_k this has to be proved by other means. The study of the eigenfunctions and eigenvalues of equations like (18) is called *Sturm-Liouville* theory.