Completing a Metric Space

MA 466
Kurt Bryan

Introduction

Recall that a metric space M is said to be *complete* if every Cauchy sequence in M converges to a limit in M. Not all metric spaces are complete, but it is a fact that all metric spaces can be "completed", in a way that preserves the essential structure of the metric space. If the space in question is a normed linear space this process completes the space to a Banach space, and an inner product space is completed to a Hilbert space.

The Space of Cauchy Sequences

Consider a metric space M with metric d, and suppose that M is not complete. Let S denote the set of all Cauchy sequences in M. We're going to turn S into a metric space in its own right, one that naturally contains a copy of M.

Let $X = \{x_n\}$ and $Y = \{y_n\}$ be elements of S (whether the sequences in question have limits in M is irrelevant). We will say that the sequences X and Y are "equivalent" if

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{1}$$

For example, if M denotes the rational numbers with the usual metric $d(x, y) = |x - y|$, let

$$X = \{2/1, 3/2, 4/3, 5/4, \ldots\}, \quad Y = \{3/1, 4/2, 5/3, 6/4, \ldots\}.$$

Both sequences are Cauchy, and moreover since $x_n = 1 + 1/n$ and $y_n = 1 + 2/n$, the nth terms differ in magnitude by $1/n$ and so $d(x_n, y_n) \to 0$ and the sequences are equivalent. That both sequences have limits in M is irrelevant.

It's easy to check that the notion of equivalence defined by equation (1) is in fact an equivalence relation, that is, any element X of S is equivalent to itself (the reflexive property), X equivalent to Y implies Y is equivalent to X (symmetric) and X equivalent to Y and Y equivalent to Z implies X equivalent to Z (transitivity).

An "Almost" Metric

Define a function Δ from $S \times S$ to $\mathbb{R}$ as

$$\Delta(X, Y) = \lim_{n\to\infty} d(x_n, y_n). \tag{2}$$

The function Δ is supposed to be a "first-stab" at a metric on S, but there are a couple of issues. First, does the limit on the right in equation (2) exist? Second, will Δ really be a metric?

To answer the first question, note that by the triangle inequality we have

$$d(x_n, y_n) \le d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n). \tag{3}$$

From equation (3) we have $d(x_n, y_n) - d(x_m, y_m) \le d(x_n, x_m) + d(y_m, y_n)$, and reversing the roles of m and n (and using symmetry of the metric) we find that

$$|d(x_n, y_n) - d(x_m, y_m)| \le d(x_n, x_m) + d(y_m, y_n). \tag{4}$$

Since both sequences are Cauchy, for any $\epsilon > 0$ we can choose N_X so that $d(x_n, x_m) < \epsilon/2$ for $m, n \ge N_x$, and also N_Y so that $d(y_n, y_m) < \epsilon/2$ for $m, n \ge N_y$. Let $N = \max(N_X, N_Y)$. From equation (4) we have then have $|d(x_n, y_n) - d(x_m, y_m)| < \epsilon$ for all $m, n \ge N$, i.e., $\{d(x_n, y_n)\}$ is a Cauchy sequence in $\mathbb{R}$, and hence the limit in equation (2) converges, since $\mathbb{R}$ is complete.

Ok, so Δ is well defined on $S \times S$—but is it a metric? It certainly satisfies $\Delta \ge 0$ and $\Delta(X, Y) = \Delta(Y, X)$. The triangle inequality follows from the triangle inequality for d, for

$$\Delta(X, Y) = \lim_{n\to\infty} d(x_n, y_n) \le \lim_{n\to\infty} d(x_n, z_n) + \lim_{n\to\infty} d(z_n, y_n) = \Delta(X, Z) + \Delta(Z, Y)$$

where $Z = \{z_n\}$ is any other Cauchy sequence. But unfortunately $\Delta(X, Y) = 0$ (i.e., X and Y are equivalent) doesn't imply that $X = Y$; the example above shows that.

Fixing the Metric

What we do to overcome this shortcoming is divide S up into its equivalence classes via the equivalence relation (1). Let M^* denote the set of equivalence classes of S; we'll use X^* and Y^* to denote typical elements of M^*; note that if $X^* \in M^*$ then X^* is a subset of S, consisting elements (Cauchy sequences in M) which are all equivalent to each other. We will define a metric d^* on M^*, as

$$d^*(X^*, Y^*) = \Delta(X, Y) \tag{5}$$

where X is any element in X^* and Y any element in Y^*. The function d^* is well-defined, independent of the choice of X and Y. To see this, let $X_2 \in X^*$ and $Y_2 \in Y^*$. Then

$$\Delta(X, Y) \leq \Delta(X, X_2) + \Delta(X_2, Y_2) + \Delta(Y_2, Y) = \Delta(X_2, Y_2)$$

since $\Delta(X, X_2) = \Delta(Y, Y_2) = 0$, so that $\Delta(X, Y) \leq \Delta(X_2, Y_2)$. A similar argument also shows $\Delta(X_2, Y_2) \leq \Delta(X, Y)$ so that $\Delta(X, Y) = \Delta(X_2, Y_2)$. The right side of equation (5) is thus independent of the choice of X and Y, and so d^* is well-defined.

We now find that if $d^*(X^*, Y^*) = 0$ then $\Delta(X, Y) = 0$ for any $X \in X^*$, $Y \in Y^*$, so that X^* and Y^* must be the same equivalence class, i.e., $X^* = Y^*$. The work we did above with Δ shows that d^* also satisfies the other properties required of a metric. The space M^* is indeed a metric space.

In general, to carry out any computation involving X^* and Y^* in M^*, we choose representatives X and Y (Cauchy sequences) in the appropriate equivalence classes and do the computation with these representatives. We then justify that the specific choices for X and Y didn't matter.

The space M^* very naturally contains a copy of the original metric space M. Specifically, for any $x \in M$, the sequence $(x, x, x, \ldots)$ is in S, and hence belongs to an equivalence class X^* in M^*. Moreover, if $y \in M$ corresponds to $(y, y, y, \ldots)$ in S and to Y^* in M^* then we find that $d^*(X^*, Y^*) = d(x, y)$. The new metric corresponds to the old if we identify each element of M with the equivalence class of its "constant" sequence. Put more mathematically, the mapping

$$\phi : y \to (y, y, y, \ldots)^*$$

from M to M^* is an isometry (distance preserving map), where the "$*$" means "equivalence class of".

Density

The original metric space M (or more appropriately, the image $\phi(M)$ of M under the isometry $\phi : y \to (y, y, y, \ldots)^*$) is dense in M^*. This is easy to prove: let Y^* be an element of M^*, and $Y \in S$ any element of Y^* (i.e., a representative for the equivalence class Y^*). Y is itself a Cauchy sequence in M. Suppose

$$Y = \{y_1, y_2, y_3, \ldots\}$$

Let $Y_k = \phi(y_k)$. It's not hard to see that Y_k converges to Y, so that $Y_k^* \to Y^*$ (and note that $Y_k^* \in \phi(M)$).

Is M^* Complete?

We've constructed a new metric space M^* with metric d^*, and M lives naturally inside M^*—but is M^* complete? Yes! But be warned, the argument, though not technically difficult, is a bit abstract. We have to consider Cauchy sequences of M^*, that is, Cauchy sequences of equivalence classes of Cauchy sequences in M!

To show M^* is complete, suppose we have a Cauchy sequence X_k^* in M^*; we need to find an element of M^* to which X_k^* converges. For each k choose a representative $X_k \in X_k^*$, and let

$$X_k = \{x_{k1}, x_{k2}, x_{k3}, \ldots, x_{kj}, \ldots\}$$

(note the x_{kj} are elements of the original metric space M). Note that to say that X_k^* is Cauchy means that for any $\epsilon > 0$ we can find some R such that $\Delta(X_m, X_n) < \epsilon$ for $m, n \geq R$, i.e.,

$$\lim_{j \to \infty} d(x_{mj}, x_{nj}) < \epsilon \tag{6}$$

for all $m, n \geq R$.

Now we'll use a kind of diagonalization argument. Since for each fixed k the sequence $\{x_{k1}, x_{k2}, x_{k3}, \ldots, x_{kj}, \ldots\}$ is Cauchy in M (with respect to the second index j) we can find some N_k such that $d(x_{kp}, x_{kq}) < 1/k$ for $p, q \geq N_k$. Choose any x_{kj} with $j \geq N_k$ and call that element y_k. We then have

$$d(y_k, x_{kj}) < 1/k \tag{7}$$

for $j \geq N_k$. For each k let Y_k denote the constant sequence

$$Y_k = (y_k, y_k, y_k, \ldots) \tag{8}$$

which is clearly Cauchy (and so $Y_k \in S$). Let Y_k^* denote the equivalence class to which Y_k belongs in M^*. An immediate consequence of inequality (7) is that $\Delta(X_k, Y_k) \leq 1/k$, and hence

$$d^*(X_k^*, Y_k^*) \leq 1/k. \tag{9}$$

Given the last inequality, if we can find a limit for Y_k^* in M^* then X_k^* will converge to the same limit.

A limit for Y_k^* isn't too hard. Let

$$Y = \{y_1, y_2, y_3, \ldots\} \tag{10}$$

The sequence $\{y_1, y_2, y_3, \ldots\}$ is Cauchy. To see this, note that from the triangle inequality we have

$$d(y_m, y_n) \leq d(y_m, x_{mj}) + d(x_{mj}, x_{nj}) + d(x_{nj}, y_n). \tag{11}$$

From equation (7) we can choose some M_2 large enough so that $d(y_m, x_{mj}) < \epsilon/3$ and $d(x_{nj}, y_n) < \epsilon/3$ for $m, n \geq M_2$ and for all j sufficiently large. From equation (6) we can also, by increasing the value of M_2 if necessary, guarantee that $d(x_{mj}, x_{nj}) < \epsilon/3$ by taking j sufficiently large. As a result we find from inequality (11) that $d(y_m, y_n) < \epsilon$ for $m, n \geq M_2$ and so $\{y_m\}$ is Cauchy.

Thus Y as defined by equation (10) is a Cauchy sequence in M and so belongs to S. It's also obvious that Y_k defined by equation (8) converges to Y (since $\Delta(Y_k, Y) = \lim_j d(y_k, y_j)$; since $\{y_j\}$ is Cauchy, $d(y_k, y_j)$ can be made small by taking j, k large), and so Y_k^* converges to Y^* where Y^* denotes the equivalence class for Y in M^*. From equation (9) we conclude that $X_k^* \to Y^* \in M^*$.

The metric space M^* is called the *completion* of M.

Banach and Hilbert Spaces

As we've seen, any inner product space is a normed linear space, and any normed linear space is a metric space. We can thus carry out this completion procedure. For a normed linear space we find that the completion is itself a normed linear space, i.e., and Banach space. Moreover, the mapping ϕ becomes an isometric isomorphism from M onto $\phi(M)$—a distance preserving map that also preserves algebraic structure, e.g, $\phi(x + y) = \phi(x) + \phi(y)$. In the case of an inner product space we end up with a Hilbert space, and the inner product is also preserved in a natural way.

Exercises:

1. Let $M = \{1, 1/2, 1/3, 1/4, \ldots\} \subset \mathbb{R}$. We can consider M to be a metric space with the usual norm $d(x, y) = |x - y|$. But S is not a complete metric space.

   Let's set $a_n = 1/n$.

   (a) Specify a Cauchy sequence in M which has no limit in M.

   (b) Show that $d(a_n, x) \geq \frac{1}{n^2+n}$ for any $x \in M$ with $x \neq a_n$.

   (c) Show that the only Cauchy sequences in M are those sequences x_n which are of

   - Type 1: Eventually constant (so $x_n = a_r$ for some r and all $n \geq N$) or;
   - Type 2: Sequences such that for each $R > 0$ there exists some $N > 0$ such that for each $n \geq N$ we have $x_n = a_r$ for some $r \geq R$. Here r may depend on n.

   (d) The completion of M consists of the equivalence classes of Cauchy sequences in M. A type 1 equivalence class corresponds to an element in already in M. What's the natural interpretation of the type 2 sequence? Hint: it's a real number.

2. Show that any uniformly continuous function T from a metric space M to a complete metric space M_2 can be extended to a continuous mapping from M^* to M_2. Hint: any point $x^* \in M^*$ is a limit of a sequence x_k in M. Use continuity to define the extension.

3. Give an example to show that the word "uniformly" in the last problem cannot be omitted. Hint: $M = (0, 1)$.

4. Suppose that we try to complete a metric space M that is already complete. Show that in this case ϕ is an invertible map that yields a one-to-one correspondence between M and M^*.

5. Is the completion of a metric space unique? Yes, up to isomorphism. To see this let M be a metric space and M^* and M^{**} be complete metric spaces with metrics d^* and d^{**} such that there exists isometric maps

ϕ_1 and ϕ_2 such that $\phi_1(M)$ is dense in M^* and $\phi_2(M)$ is dense in M^{**}. Show that there is an isometric one-to-one map ψ from M^* onto M^{**} with $\psi(\phi_1(x)) = \phi_2(x)$ for all $x \in M$, and $d^{**}(\psi(x), \psi(y)) = d^*(x, y)$ for all $x, y \in M^*$. Thus up to the mapping ψ, $M^* = M^{**}$