Compactness and Continuity

MA 466
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Let M_1 be a metric space with metric d_1 and M_2 a metric space with metric d_2. Recall that a function $f : M_1 \to M_2$ is continuous at $x \in M_1$ if for each $\epsilon > 0$ there is a $\delta > 0$ such that $d_2(f(x), f(y)) < \epsilon$ for all y such that $d_1(x, y) < \delta$.

Here's a useful lemma concerning continuity.

Lemma 1 *A function f from M_1 to M_2 is continuous at a point $x \in M_1$ if and only if*

$$\lim_{n\to\infty} f(x_n) = f(x)$$

for every sequence x_n which converges to x.

Proof: Suppose f is continuous at x and x_n is any sequence which converges to x. Let $\epsilon > 0$ be given and take δ as in the definition of continuity. Since $x_n \to x$ we can choose some N such that $d_1(x_n, x) < \delta$ for all $n \geq N$, and this means that $d_2(f(x), f(x_n)) < \epsilon$ for all $n \geq N$. Thus $f(x_n)$ converges to $f(x)$.

Conversely, suppose that $f(x_n)$ converges to $f(x)$ for all sequences x_n which converge to x, but suppose that f is NOT continuous at x. The latter means that we can find some $\epsilon > 0$ and some $x_n \in M_1$ such that $d_2(f(x), f(x_n)) \geq \epsilon$ and yet $d_1(x_n, x) < 1/n$. But this implies a clear contradiction, for then $x_n \to x$ but $f(x_n)$ cannot converge to $f(x)$. $\Box$

A function f from $E \subset M_1$ to M_2 is bounded if the set $S = \{f(x)|x \in E\}$ is bounded in M_2, that is, $S \subset B_r(b)$ for some $b \in M_2$ and some $r > 0$. Recall that the choice of b in the definition of bounded doesn't matter—if S is bounded using one choice for b, S is bounded for any other.

Another very useful theorem from reals is this.

Theorem 1 *Let K be a compact subset of metric space M_1 and $f : K \to M_2$ a continuous function on K. Then the image $f(K)$ is compact in M_2.*

Proof: Let y_k be a sequence contained in $f(K)$. Each $y_k = f(x_k)$ for some $x_k \in K$. Now since K is compact we can choose a subsequence x_{k_j} which converges to some point $x^* \in K$. From the continuity of f we conclude that the subsequence $y_{k_j} f = (x_{k_j})$ converges to $y^* = f(x^*) \in f(K)$, so that any sequence in $f(K)$ has a convergent subsequence. Thus $f(K)$ is compact. $\square$

Of course this means that $f(K)$ is closed and bounded. The latter shows that a continuous function on a compact set is necessarily bounded.

Here's a nice fact from reals that also extends:

Theorem 2 *Let K be a compact subset of metric space M_1 and $f : K \to \mathbb{R}$ a continuous function on K. Then f attains it's maximum value at some point in K.*

Proof: We know from the previous theorem that f is bounded; let $M = \sup_{x\in K} f(x)$. We'll show that $f(a) = M$ for some point $a \in K$.

To see this, choose a sequence $x_n \in K$ such that $M - 1/n < f(x_n) \le M$. Since K is compact, we can extract a subsequence x_{n_k} which converges to some point $a \in K$. Since f is continuous we have, by Lemma 1,

$$f(a) = \lim_{n_k\to\infty} f(x).$$

But the right side above clearly converges to M, so $f(a) = M$. $\square$

Here's a little something to think about: Why is f in the last theorem required to have $\mathbb{R}$ as its range, instead some arbitrary metric space?

Let $E \subset M_1$. A function f defined on E is continuous on E if f is continuous at each point in E. A function f is *uniformly continuous* on E if f is continuous at each $x \in E$ and for any ϵ in the continuity definition the δ can be chosen independently of x. In short, for any given ϵ we can choose δ in a "one size fits all x" fashion.

Here's a theorem from real analysis that generalizes to the present setting.

Theorem 3 *Let K be a compact subset of metric space M_1 and $f : K \to M_2$ a continuous function on K. Then f is uniformly continuous.*

Proof: We'll do a proof by contradiction: Suppose that f is not uniformly continuous. Then for any $\epsilon > 0$ we can find points x_n and y_n in K such that $d_1(x_n, y_n) < 1/n$ but $d_2(f(x_n), f(y_n)) \ge \epsilon$. Since K is compact we can extract subsequences x_{n_k} and y_{n_k} which converge to points in K; indeed, both subsequences must converge to the same point a, since $d_1(x_{n_k}, y_{n_k}) < 1/n_k$. Since f is continuous at each point in K we must have (by Lemma 1)

$$\lim_{k\to\infty} f(x_{n_k}) = f(a) = \lim_{k\to\infty} f(y_{n_k})$$

which is clearly impossible since $d_2(f(x_{n_k}), f(y_{n_k})) \ge \epsilon$. We conclude that f must be uniformly continuous. $\square$