More On Conservation of Stuff

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Introduction

Yesterday we arrived at the relation

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{1}$$

for stuff flowing through a conduit. The function $\rho(x,t)$ is the density of the stuff at the point x at time t and $q(x,t)$ is the rate that stuff flows past the point x at time t. The above equation was a direct consequence of the assumption that the total amount of stuff is conserved; none is created or destroyed. This equation is extremely general and applies to many physical situations, from cars on a road to electrons in a wire. It's called the *continuity equation*.

Our ultimate goal is to solve for either ρ or q and thus say something about what's going on physically. The problem is that we have only one equation and two unknowns, ρ and q. We need a second equation relating the two functions, a so-called *constitutive relation* between ρ and q. This equation can't be pulled out of thin air, though. We need to use specific information about the phenomena we're trying to model. This is where cars and electrons part company. We'll now examine several different situations, arriving at different models for each.

Advection

Let's begin with the simplest situation. The function $\rho(x,t)$ represents the concentration of pollutants in a river; ρ has units of kilograms of pollutant per meter of river length. The width of the river is assumed to be negligible in comparison to its length. In fact, suppose the river is infinitely long, so $-\infty < x < \infty$. The pollutants are carried downstream by the river and don't tend to spread out ("diffuse") on their own. Let's suppose that the speed of the river water is constant and equal to c meters per second.

With this information we can find a relation between ρ and q. Imagine a line L drawn straight across the river. From time t to time $t + dt$ a certain amount of water flows past L. Now since we're thinking of the river as being of uniform cross-section, we can specify this amount of water as a "length"

(whose product with the river's cross sectional area would give the volume that flowed past). The amount of water flowing past in time dt is in fact $c\,dt$ meters, and so the amount of pollutant contained in this water is $c\rho(x,t)\,dt$ kg (if dt is small, so ρ will be roughly constant). Since the amount of pollutant flowing past L from t to $t+dt$ is $c\rho(x,t)\,dt$, the rate at which pollutant flows past L is the amount over dt, i.e., that rate is $c\rho(x,t)$. But this rate is also by definition $q(x,t)$. So the relation between ρ and q is

$$q(x,t) = c\rho(x,t). \tag{2}$$

Exercise

- Check that equation (2) balances with respect to dimension!

Combining equation (2) with the continuity equation (1) leads to

$$\frac{\partial \rho}{\partial t} + c\frac{\partial \rho}{\partial x} = 0, \tag{3}$$

the so-called *advection* or *one-way wave* equation. It is a partial differential equation that the function $\rho(x,t)$ must satisfy, given the physics of the situation. The constant c is the *wave speed*.

Thinking Problems

- Pretend the river is infinitely long, flows from left to right at speed c, and that at time $t = 0$ the concentration of pollutants $\rho(x,0)$ looks like

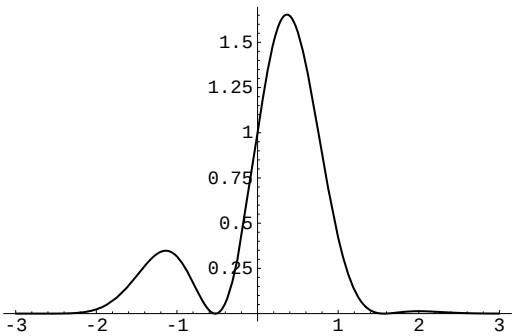

Figure 1

for x in $[-3,3]$. Assume that the pollutants have zero concentration

outside the picture. What would the pollutant concentration look like at a later time? If it helps, take specific values for c and t, say $c = 2$ meters per second and $t = 3$.

- Suppose that the initial pollution concentration at time $t = 0$ pictured above is described by some function $f(x)$, that is, $\rho(x, 0) = f(x)$. What is a formula (in terms of f) for $\rho(x, t)$ at any later time?

The solution above is called a *travelling wave*. It's simply some initial "shape" function propagated downstream at speed c. Notice that the solution $\rho(x, t)$ is constant for any fixed value of $x - ct$. That is, $\rho(x, t)$ has a constant value on lines of the form $x - ct = b$, where b is any constant. These lines are called the *characteristics* of the differential equation. A picture of the solution over time and characteristics are given below.

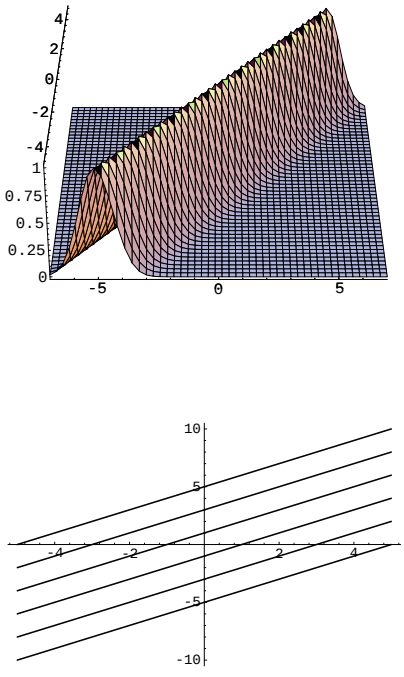

Figure 2

Problems

- If the solution $u(x,t)$ to $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$ looks like Figure 1 above at time $t = 0$, estimate

  1. $u(3,3)$
  2. $u(4,1)$
  3. $u(3/2,1)$.

More on Characteristics

There's another way to see that solutions to equation (3) must be constant on characteristics $x - ct = b$, without actually solving the DE. Consider a solution $\rho(x,t)$ on such a characteristic, so $x = b + ct$. Now

$$\begin{aligned}\frac{d}{dt}(\rho(b+ct,t)) &= c\frac{\partial \rho}{\partial x}(b+ct,t) + \frac{\partial \rho}{\partial t}(b+ct,t) \\ &= c\frac{\partial \rho}{\partial x}(b+ct,t) - c\frac{\partial \rho}{\partial x}(b+ct,t) \\ &= 0\end{aligned}$$

where I've used the chain rule and the fact that $\frac{\partial \rho}{\partial t} = -c\frac{\partial \rho}{\partial x}$. Since $\frac{d}{dt}(\rho(b+ct,t)) \equiv 0$ we conclude that $\rho(b+ct,t)$ doesn't depend on t, and hence cannot vary along the characteristic (because as t varies, $(b+ct,t)$ indexes every point on the characteristic).

In Summary

In summary, to solve the DE $\frac{\partial \rho}{\partial t} + c\frac{\partial \rho}{\partial x} = 0$ for $-\infty < x < \infty$ and $t > 0$ with $\rho(x,0) = f(x)$, do the following: Pick some point (x_0, t_0) at which you'd like to compute ρ. The point (x_0, t_0) lies on some characteristic $x - ct = b$, where clearly $b = x_0 - ct_0$, so the characteristic is just $x - ct = x_0 - ct_0$. Follow the characteristic back to the initial condition at $t = 0$, so $x = x_0 - ct_0$. Since the solution is constant on the characteristics, $\rho(x_0, t_0) = \rho(x_0 - ct_0, 0) = f(x_0 - ct_0)$.