Euclidean n-space

MA 466
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Maybe you didn't see it when you took real analysis, but $\mathbb{R}^n$ with the usual inner product is a Hilbert space, that is, an inner product space which is complete. The completeness of $\mathbb{R}^n$ follows pretty simply from the fact that $\mathbb{R}$ is complete, which I'll show below.

Recall that if $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ then the inner product is of course just

$$< x, y >= \sum_{k=1}^{n} x_k y_k, \tag{1}$$

the "dot product". The induced norm is

$$\|x\|_2 = \sqrt{< x, x >} = \left(\sum_{k=1}^{n} x_k^2 \right)^{1/2} \tag{2}$$

and of course the corresponding metric is

$$d(x, y) = \|x - y\|_2 = \left(\sum_{k=1}^{n} (x_k - y_k)^2 \right)^{1/2}, \tag{3}$$

the usual distance formula. We've already checked that $< x, y >$ as defined is an inner product; we'll show now that with the induced metric $\mathbb{R}^n$ is in fact complete.

In what follows I'll use superscripts to index the elements of a sequence, so x^k refers to the kth element in a sequence of vectors in $\mathbb{R}^n$. I'll write x_j^k to refer to the jth component of the kth element.

Theorem 1: A sequence $\{x^k\}$ in $\mathbb{R}^n$ converges to a vector $x = (x_1, \ldots, x_n)$ if and only if each of the sequences $\{x_j^k\}$, $1 \leq j \leq n$, converges to x_j in $\mathbb{R}$.

Proof: Suppose that the sequence x^k converges to the vector $x = (x_1, \ldots, x_n)$. Then for any $\epsilon > 0$ we can find some N such that $d(x^k, x) < \epsilon$ for all $k \geq N$. It's easy to see from (3) that this forces $|x_j^k - x_j| < \epsilon$ too, so we conclude that x_j^k converges to x_j in $\mathbb{R}$.

Conversely, suppose that each of the sequences $\{x_j^k\}$, $1 \leq j \leq n$, converge to some number x_j. Then for any $\epsilon > 0$ we can find some N_j such that

$$|x_j^k - x_j| < \epsilon/\sqrt{n}$$

for $k \geq N_j$. Let $N = \max_j N_j$ (the maximum must exist!) For $k \geq N$ we have $|x_j^k - x_j| < \epsilon/\sqrt{n}$ for $1 \leq j \leq n$. It follows easily from equation (3) that $d(x^k, x) < \epsilon$, where $x = (x_1, \ldots, x_n)$, so x^k converges to x in $\mathbb{R}^n$. $\Box$

So a sequence converges in $\mathbb{R}^n$ iff it converges component by component. The same fact is true concerning the Cauchy criterium:

Theorem 2: A sequence $\{x^k\}$ in $\mathbb{R}^n$ is Cauchy if and only if each of the sequences $\{x_j^k\}$, $1 \leq j \leq n$, is Cauchy in $\mathbb{R}$.

Proof: Suppose that the sequence x^k is Cauchy in $\mathbb{R}^n$. Then for any $\epsilon > 0$ we can find some N such that $d(x^k, x^m) < \epsilon$ for $k, m \geq N$. It's easy to see from (3) that this forces $|x_j^k - x_j^m| < \epsilon$ too, so we conclude that x_j^k is Cauchy in $\mathbb{R}$.

Conversely, suppose that each of the sequences $\{x_j^k\}$, $1 \leq j \leq n$, is Cauchy in $\mathbb{R}$. Then for any $\epsilon > 0$ we can find some N_j such that

$$|x_j^k - x_j^m| < \epsilon/\sqrt{n}$$

for $k, m \geq N_j$. Let $N = \max_j N_j$ (the maximum must exist!) For $k, m \geq N$ we have $|x_j^k - x_j^m| < \epsilon/\sqrt{n}$ for $1 \leq j \leq n$. It follows easily from equation (3) that $d(x^k, x^m) < \epsilon$ so that x^k is Cauchy in $\mathbb{R}^n$.

Theorem 3: The space $\mathbb{R}^n$ with the inner product (1) is complete.

Proof: Let x^k be a Cauchy sequence in $\mathbb{R}^n$. By Theorem 2 each of the sequences x_j^k, $1 \leq j \leq n$, is Cauchy in $\mathbb{R}$ and hence converges to some number, say x_j. By Theorem 1 the sequence x^k converges to the vector $x = (x_1, \ldots, x_n)$. Since each Cauchy sequence in $\mathbb{R}^n$ converges, $\mathbb{R}^n$ is complete.