Higher Genus Soccer Balls

S. Allen Broughton

Rose-Hulman Institute of Technology
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Outline

1. Introduction/Background
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   - Why soccer balls?
   - Kaleidoscopic tilings

2. Tiling to soccer ball
   - Cayley Graph Construction
   - Hyperbolic geometry and surfaces

3. Group theory
   - Tiling groups
   - Results

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   - Classification
   - Representation
most work done with undergraduates in RHIT NSF-REU
research site for the work http://www.tilings.org/
also joint work with Aaron Wooton at University of Portland
why soccer balls - 1

- Here is a picture of a soccer ball borrowed from everbe.com via Google.

- observe that the pentagons and hexagons are regular
- there are twelve pentagons and twenty hexagons
Why soccer balls?

**why soccer balls - 2**

- motivation from chemistry
  - soccer ball is a model for bucky balls and nanotubes
  - see pictures next two slides
- motivation from fibre arts
  - construction of Tamari balls, see later slides
  - baseball caps - questions from a cap maker
why soccer balls - 3

- Here is a ball and stick model of a bucky ball from the site http://www.psyclops.com/bucky.shtml

- There are 60 atoms and 90 bonds
- Chemists observe that the pentagons are regular but the hexagons are not
Here is a cartoon model of a small single walled carbon nanotube (SWT) from the site http://www.icpf.cas.cz/jiri/pictures/nanotube.jpg

there are 12 pentagons and some number of hexagons
why soccer balls - 5

Here is a picture of some Tamari Balls borrowed from http://www.istockphoto.com/
consider tiling by (2,3,5)-triangles on the sphere

reflections in the edges of triangles preserve the tiling
kaleidoscopic tiling - definition

- $S$ is a surface of genus $\sigma$ (assume the sphere for the moment)
- Tiling: covering of $S$ by polygons “without gaps and overlaps”
- Kaleidoscopic: symmetric via reflections in edges
- Geodesic: edges in tiles extend to geodesics in both directions
 Terminology: \((\ell, m, n)\)-triangle, angles are \(\pi/\ell, \pi/m, \pi/n\)
 kaleidoscopic triangles on the sphere \((2, 2, d), (2, 3, 3), (2, 3, 4), (2, 3, 5)\)
 we will see these in a moment when we visit the soccer ball page
kaleidoscopic tilings - number of triangles

Let $S$ be a surface of genus $\sigma$ and $2|G|$ the number of triangles

then

$$\frac{2\sigma - 2}{|G|} = 1 - \frac{1}{\ell} - \frac{1}{m} - \frac{1}{n}$$
Cayley Graph construction - 1

- select a distinguished tile called the master tile.
- select a point in master tile
- reflect the point in three sides of the triangle
- join the new points to the original point by line segments
- repeat until no more new points
Cayley Graph construction - 2

- example for (2, 3, 5) tiling

next visit soccer ball page http://www.rose-hulman.edu/ brought/Epubs/soccer/soccer.html

during visit observe that torus examples are derived from tilings and Cayley graphs on the plane
Hyperbolic geometry and surfaces

Hyperbolic geometry - 1

- show big picture: t433.pdf
from the big picture
points: interior points of unit circle
lines: circles perpendicular to boundary of the unit circle or a diameter
angles: angle of intersection via calculus
distance: ghastly formula
Reflections: inversion in a circle

\[ |z - O| |z' - O| = r^2 \]
for hyperbolic triangles sum of angles less than 180 so

$$\frac{\pi}{\ell} + \frac{\pi}{m} + \frac{\pi}{n} < \pi$$

both sides of following equation are positive when $\sigma > 2$

$$\frac{2\sigma - 2}{|G|} = 1 - \frac{1}{\ell} - \frac{1}{m} - \frac{1}{n}$$
Hyperbolic surfaces

- the surfaces with the tilings are not easily representable in a "geometrically faithful" way in three dimensions
- as in the case of the torus represent as a finite collection of tiles suitably rolled up
- see the further questions section
Tiling groups

**tiling group construction - 1**

- $G^*$ group generated by reflections in edges of a tile - called a reflection group
- $G$ is the subgroup of group $G^*$ which are orientation preserving - called a rotation group
- describe group by means of the following picture
define

\[ a = pq, \quad b = qr, \quad c = rp \]

\[ G^* = \langle p, q, r \rangle \]
\[ G = \langle a, b, c \rangle \]

\[ p^2 = q^2 = r^2 = 1 \]
\[ a^\ell = b^m = c^n = abc = 1 \]

\[ \theta(a) = qaq = qpqq = qp = a^{-1} \]
\[ \theta(b) = qbq = qqrq = rp = b^{-1} \]
find

\[ G = \langle a, b, c | a^\ell = b^m = c^n = abc = 1 \rangle \]

and an automorphism \( \theta \)

\[ \theta(a) = qaq = qpqq = qp = a^{-1} \]
\[ \theta(b) = qbq = qqrq = rp = b^{-1} \]

always true for abelian groups

frequently true for general groups
all kaleidoscopic triangular tilings up to genus 25 have been determined

focus of REU program - how I met Robert J.
an investigation by quadrilaterals has been started
all kaleidoscopic quadrangular tilings up to genus 13 have been determined (REU program)
with Aaron Wooton some general results on abelian groups have been obtained
all kaleidoscopic quadrilateral tilings with $G = \text{elementary abelian group}$ have been determined.
Further questions - classification

- complete classification of all tilings for low genus (up to genus 50) is doable
- you need to use a computer algebra system such as Magma or GAP
except for one nice sculpture by Ferguson there are few pictures of a 3D representation of a genus 2 or greater tiling or "soccer ball"

especially find some nice pictures of “Platonic” Cayley graphs in which there is only one regular polygon

at least one such tiling arises as the configuration space of a physical system

find a “nice set” of tiles or polygons which represents the surface after identification of sides